



Implicit differentiation

Year 12 Specialist Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what an implicit equation is
- Understand how to do implicit differentiation



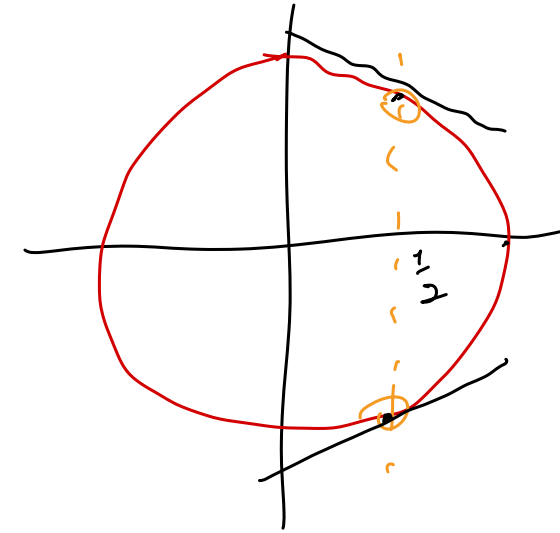
Recap of past learning

A circle is an example of an implicit function. This means it's written in terms of both dependent and independent variables. A good example is shown below:

$$x^2 + y^2 = 1$$

The circle cannot be expressed in the form $y = f(x)$ or $x = f(y)$ as they are not functions.

As a result, it would be good if we could find a way to differentiate these equations without having to split it into two separate equations.



A different way of doing things ...

We have already met a way to differentiate the following:

$$x = y^3$$

By doing $\frac{dx}{dy}$ and then taking the reciprocal.

We now look at another way of doing it.

$$x = y^3$$

$$\frac{dx}{dy} = 3y^2$$

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

$$\frac{dy}{dx} = \frac{1}{3y^2}$$

$$x \rightarrow y$$

$$y = \sqrt[3]{x}$$

$$= x^{1/3}$$

$$= \frac{1}{3} x^{-2/3}$$

$$= \frac{1}{3\sqrt[3]{x^2}}$$



Implicit differentiation

Let's refactor the question

$$\frac{d}{dx}(x) = \frac{d}{dx}(y^3)$$

Now, using the chain rule let $u = y^3$...

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$$

We know that $\frac{du}{dy} = 3y^2$

$$\text{So, } \frac{du}{dx} = 3y^2 \times \frac{dy}{dx}$$

$$\frac{du}{dx} = 3y^2 \times \frac{dy}{dx}$$

$$x = y^3$$
$$1 = 3y^2 \cdot \frac{dy}{dx}$$

This is really important! It means you simply differentiate the y term and multiply it by $\frac{dy}{dx}$

Implicit differentiation is used when you have an equation where y is not explicitly isolated as a function of x ($y = f(x)$), and you need to find dy/dx . It's particularly useful for equations that are difficult or impossible to solve for y directly

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$$

Substituting back gives:

$$1 = 3y^2 \times \frac{dy}{dx}$$

Or

$$\frac{dy}{dx} = \frac{1}{3y^2}$$

This is simply the differential of x

$$\frac{d(y^3)}{dx} = 3y^2 \cdot \frac{dy}{dx}$$



So much work!

Seems like a lot of hard work ... but I'm going to assume it's going to be helpful for us later on!

Example: For each of the following, find $\frac{dy}{dx}$ by implicit differentiation:

- $x^3 = y^2$
- $xy = 2x + 1$

$$x^3 = y^2$$
$$y = \sqrt[+]{x^3}$$

$$x^3 = y^2$$
$$3x^2 = 2y' \cdot \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{3x^2}{2y}$$



So much work!

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Example: For each of the following, find $\frac{dy}{dx}$ by implicit differentiation:

- $x^3 = y^2$
- $xy = 2x + 1$

$$\overset{u}{x} \overset{v}{y} = 2x + 1$$

$$x \cdot \frac{dy}{dx} = 2 - y$$

$$x \cdot 1 \cdot \frac{dy}{dx} + y \cdot 1 = 2$$

$$\frac{dy}{dx} = \frac{2 - y}{x}$$

$$x \cdot \frac{dy}{dx} + y = 2$$



I'm liking this ...

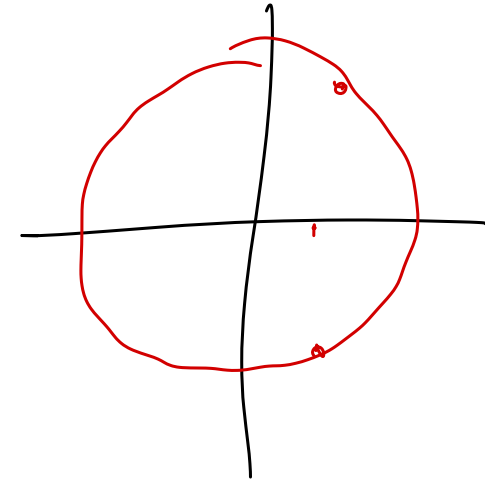
Find $\frac{dy}{dx}$ if $x^2 + y^2 = 1$ using implicit differentiation

$$x^2 + y^2 = 1$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$2y \cdot \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$



Another way ...

Find $\frac{dy}{dx}$ if $x^2 + y^2 = 1$ using ~~implicit differentiation~~

$$x^2 + y^2 = 1$$

$$y^2 = 1 - x^2$$

$$y = \pm \sqrt{1 - x^2}$$

$$\therefore y = \sqrt{1 - x^2} \quad y = -\sqrt{1 - x^2}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{x}} (1 - x^2)^{-1/2} \cdot \cancel{-2x}$$

$$\cancel{y} = \cancel{-1} (1 - x^2)^{-1/2} \cdot \cancel{-2x}$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{1 - x^2}} = -\frac{x}{\sqrt{1 - x^2}}$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{1 - x^2}} = -\frac{x}{\sqrt{1 - x^2}}$$

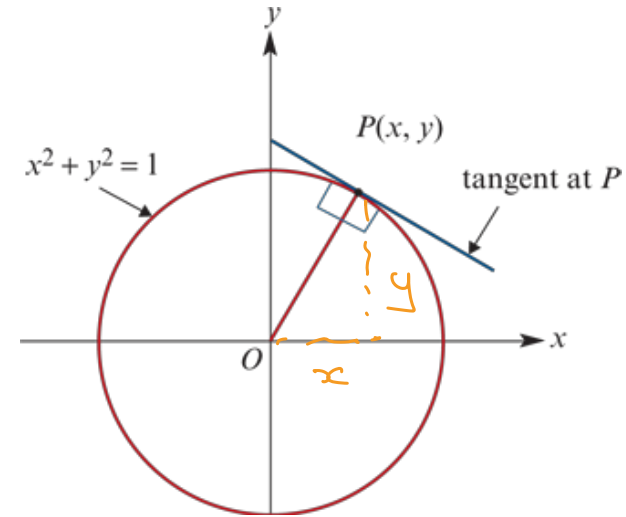


Looking geometrically at the question

Find $\frac{dy}{dx}$ if $x^2 + y^2 = 1$

Choosing a point P , we can find the gradient OP and we know the gradient of the tangent at that point can also be found.

Note: The gradient when $y = 0$ cannot be found as the gradient of the tangent would be ∞



$$m_{op} = \frac{y}{x}$$

$$m_{tan} = -\frac{x}{y}$$



Example

Given $xy - y - x^2 = 0$, find $\frac{dy}{dx}$.

$$x \cdot 1 \cdot \frac{dy}{dx} + y \cdot 1 - 1 \cdot \frac{dy}{dx} - 2x = 0$$

$$x \cdot \frac{dy}{dx} + y - \frac{dy}{dx} - 2x = 0$$

$$x \cdot \frac{dy}{dx} - \frac{dy}{dx} = 2x - y$$

$$\frac{dy}{dx} (x - 1) = 2x - y$$

$$\frac{dy}{dx} = \frac{2x - y}{x - 1}$$



Example

Consider the curve with equation $2x^2 - 2xy + y^2 = 5$.

- Find $\frac{dy}{dx}$.
- Find the gradient of the tangent to the curve at the point (1,3).

$$4x - \left(2x \cdot 1 \cdot \frac{dy}{dx} + y \cdot 2 \right) + 2y \cdot \frac{dy}{dx} = 0$$

$$4x - 2x \cdot \frac{dy}{dx} - 2y + 2y \cdot \frac{dy}{dx} = 0$$

$$y \cdot \frac{dy}{dx} - x \cdot \frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} (y - x) = y - 2x$$

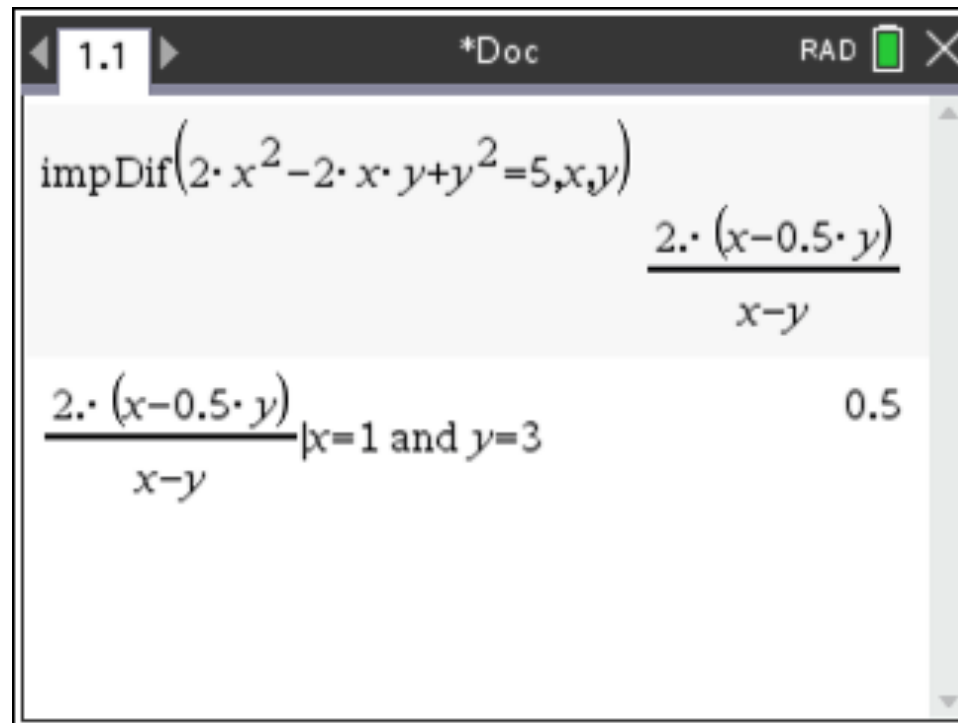
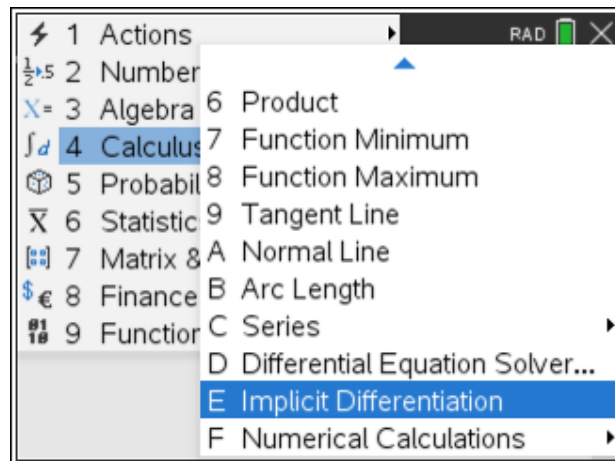
$$\frac{dy}{dx} = \frac{y - 2x}{y - x}$$

$$\therefore \frac{dy}{dx} = \frac{3 - 2}{3 - 1} = \underline{\underline{\frac{1}{2}}}$$



Using the CAS

There is the implicit differentiation function we can use!



Learning Objectives: Revisited

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