Derivatives of inverse circular functions

Year 12 Specialist Maths Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand how we can find and use the derivatives of the inverse circular functions

Recap

In previous lessons we have been looking at the basics of differentiation and then moved onto the idea of finding the derivate of $x = f(y)$.

We now move onto the much more exciting concepts of derivatives of inverse circular functions!

$x = f(y)$

Let's find the derivative of $sin^{-1}(x)$

Start with the answer and work backwards?

If $f(x) = \sin^{-1}(x)$ then $f'(x) = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1,1)$

$y = \sin^{-1}(x)$

$\sin y = x$

$x = \sin y$

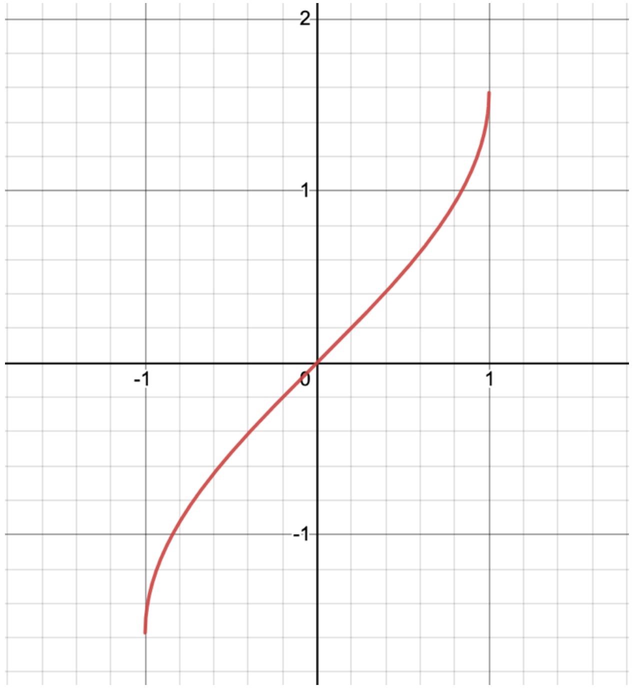
$\frac{dx}{dy} = \cos y$

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

$\sin^2 y + \cos^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \sqrt{1-\sin^2 y}$



The graph for reference

Let's find the derivative of $cos^{-1}(x)$

Start with the answer and work backwards?

If $f(x) = \cos^{-1}(x)$ then $f'(x) = \frac{-1}{\sqrt{1-x^2}}$ for $x \in (-1,1)$

$$y = \cos^{-1}(x)$$

$$\cos y = x$$

$$x = \cos y$$

$$\frac{dx}{dy} = -\sin y$$

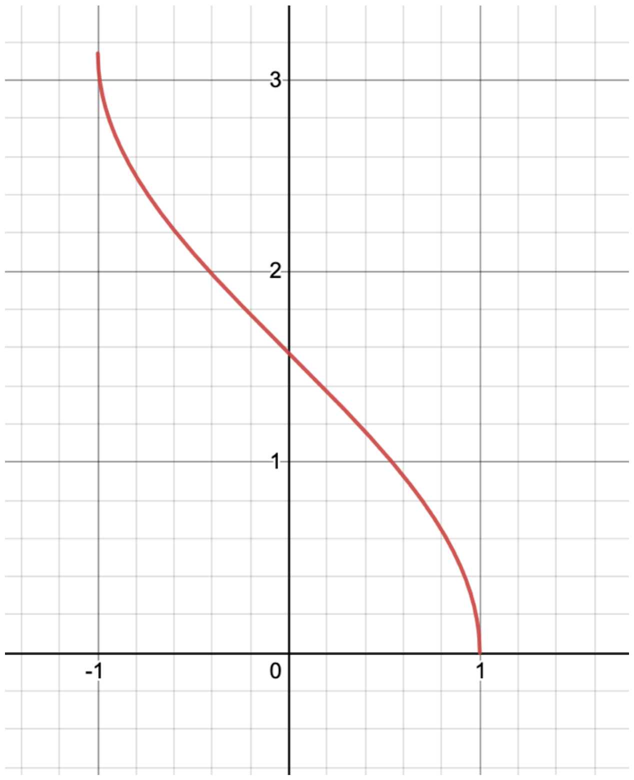
$$\frac{dy}{dx} = \frac{1}{-\sin y} = \frac{1}{-\sqrt{1-\cos^2 y}}$$

$$= \frac{-1}{\sqrt{1-x^2}}$$

$$\sin^2 y + \cos^2 y = 1$$

$$\sin^2 y = 1 - \cos^2 y$$

$$\sin y = \sqrt{1-\cos^2 y}$$



The graph for reference

Let's find the derivative of $\tan^{-1}(x)$

Start with the answer and work backwards?

If $f(x) = \tan^{-1}(x)$ then $f'(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$

$$\sec^2 y = 1 + \tan^2 y$$

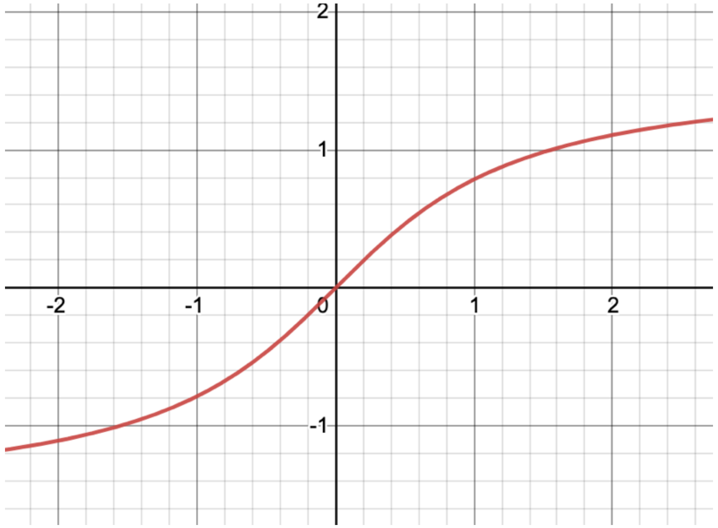
$$y = \tan^{-1}(x)$$

$$\tan y = x$$

$$x = \tan y$$

$$\frac{dx}{dy} = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$



The graph for reference

General inverse circular functions

And here they are again but for the summary book!

Inverse circular functions

$$f : (-a, a) \to \mathbb{R}, \qquad f(x) = \sin^{-1}\left(\frac{x}{a}\right), \qquad f'(x) = \frac{1}{\sqrt{a^2 - x^2}}$$

$$f : (-a, a) \to \mathbb{R}, \qquad f(x) = \cos^{-1}\left(\frac{x}{a}\right), \qquad f'(x) = \frac{-1}{\sqrt{a^2 - x^2}}$$

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \tan^{-1}\left(\frac{x}{a}\right), \qquad f'(x) = \frac{a}{a^2 + x^2}$$

Examples

Differentiate each of the following with respect to x

$\mathbf{\sin^{-1}\left(\frac{x}{3}\right)}$

$\cos^{-1}(4x)$

$\tan^{-1}\left(\frac{2x}{3}\right)$

$\sin^{-1}(x^2 - 1)$

$y = \sin^{-1}\left(\frac{x}{3}\right)$

$y' = \frac{1}{\sqrt{9 - x^2}}$

Inverse circular functions

$f: (-a, a) \to \mathbb{R}, \quad f(x) = \sin^{-1}\left(\frac{x}{a}\right), \quad f'(x) = \frac{1}{\sqrt{a^2 - x^2}}$

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$f: \mathbb{R} \to \mathbb{R}$, $f(x) = \tan^{-1}\left(\frac{x}{a}\right)$, $f'(x) = \frac{a}{a^2 + x^2}$

$y = \cos^{-1}(4x)$ $\quad u = 4x$

$y = \cos^{-1}(u)$ $\quad u' = 4$

$y' = \frac{-1}{\sqrt{1-(4x)^2}} \cdot 4$

$= \frac{-4}{\sqrt{1-16x^2}}$

$y = \cos^{-1}\left(\frac{x}{1/4}\right)$

$y' = \frac{-1}{\sqrt{\frac{1}{16} - x^2}}$ $\quad = \frac{-4}{\sqrt{1-16x^2}}$

$= \frac{-1}{\sqrt{\frac{1-16x^2}{16}}}$

$= \frac{-1}{\frac{\sqrt{1-16x^2}}{4}}$

Inverse circular functions

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$\sin^{-1}\left(\frac{x}{3}\right)$

$\cos^{-1}(4x)$

$\mathbf{\tan^{-1}\left(\frac{2x}{3}\right)}$

$\sin^{-1}(x^2 - 1)$

$$y = \tan^{-1}\left(\frac{x}{3/2}\right)$$

$$y' = \frac{\frac{3}{2}}{\frac{9}{4} + x^2}$$

$$y' = \frac{\left(\frac{3}{2}\right)}{\left(\frac{9 + 4x^2}{4}\right)} = \frac{3}{\cancel{2}} \times \frac{\overset{2}{\cancel{4}}}{9 + 4x^2}$$

$$= \frac{6}{9 + 4x^2}$$

Examples

Differentiate each of the following with respect to x

$\sin^{-1}\left(\frac{x}{3}\right)$

$\cos^{-1}(4x)$

$\mathbf{\tan^{-1}\left(\frac{2x}{3}\right)}$

$\sin^{-1}(x^2-1)$

$$y = \tan^{-1}\left(\frac{2x}{3}\right) \qquad u = \frac{2x}{3}$$

$$y = \tan^{-1}(u) \qquad \frac{du}{dx} = \frac{2}{3}$$

$$\frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{2x}{3}\right)^2} \cdot \frac{2}{3}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+\frac{4x^2}{9}} \cdot \frac{2}{3}$$

$$= \frac{1}{\frac{9+4x^2}{9}} \cdot \frac{2}{3}$$

$$= \frac{\cancel{9}\,3}{9+4x^2} \cdot \frac{2}{\cancel{3}}$$

Inverse circular functions

$f: (-a, a) \to \mathbb{R}$, $\quad f(x) = \sin^{-1}\left(\frac{x}{a}\right)$, $\quad f'(x) = \frac{1}{\sqrt{a^2 - x^2}}$

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$\cos^{-1}(4x)$

$\tan^{-1}\left(\frac{2x}{3}\right)$

$\mathbf{\sin^{-1}(x^2 - 1)}$

$y = \sin^{-1}(x^2 - 1)$ $\qquad u = x^2 - 1$

$\frac{du}{dx} = 2x$

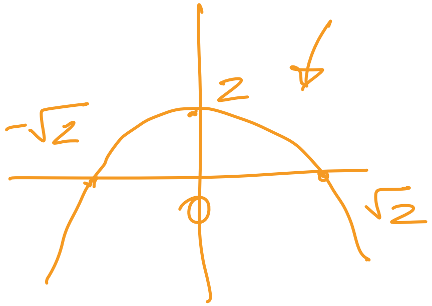
$\frac{dy}{du} = \frac{1}{\sqrt{1 - u^2}}$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1 - (x^2 - 1)^2}} \cdot 2x$

$= \frac{2x}{\sqrt{2x^2 - x^4}}$

$\frac{dy}{dx} = \frac{2x}{\sqrt{x^2}\sqrt{2 - x^2}}$

$\frac{dy}{dx} = \frac{2x}{|x|\sqrt{2 - x^2}}$



$0 < x < \sqrt{2}$ $\qquad \therefore \frac{dy}{dx} = \frac{2}{\sqrt{2 - x^2}}$

$-\sqrt{2} < x < 0$ $\qquad \therefore \frac{dy}{dx} = \frac{-2}{\sqrt{2 - x^2}}$

Learning Objectives: Revisited

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand how we can find and use the derivatives of the inverse circular functions

Work to be completed

The following represents the minimum work which should be completed.

The more questions you answer from each exercise, chapter review and Checkpoints the better you chance of gaining an excellent study score in November.

Specialist Mathematics Units 3 and 4 Textbook

Chapter 8
Exercise 8C: Derivatives of inverse circular functions
Questions: All questions

Note: I have been advised to set you all questions from Chapters 6 and 7 to ensure you are sufficiently prepared for Chapters 8, 9 and 10. The exact quote I have is, "*If they don't do them all they won't be able to access the content in those chapters and may well fail*".