Derivatives of $x=f(y)$

Year 12 Specialist Maths
Units 3 and 4

Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand how we can find the derivatives of $x = f(y)$.

Recap

In the first lesson we looked at how to differentiate and covered some important tools which we will need in our chest:

- Product Rule
- Quotient Rule
- Chain Rule

To date, we've only really ever been asked to find differentials of the equations in the form $y = f(x)$

Now we are going to look at finding the derivative of functions of the form $x = f(y)$ which may look familiar to you!

The Chain Rule revisited

We know the chain rule can be expressed in another form:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Let’s consider the special case where $y = x$

This leads us to $1 = \frac{dx}{du} \times \frac{du}{dx}$

Which seems fairly obvious! But we can use the idea then to state that:

$$\frac{du}{dx} = \frac{1}{\frac{dx}{du}}$$

Or … for our case:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

This is provided that $\frac{dx}{dy} \neq 0$

$y = x$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\frac{dx}{dx} = \frac{du}{dx} \times \frac{dx}{du}$$

$$1 = \frac{du}{dx} \times \frac{dx}{du}$$

$$\frac{du}{dx} = \frac{1}{dx/du}$$

Examples of when this would be used

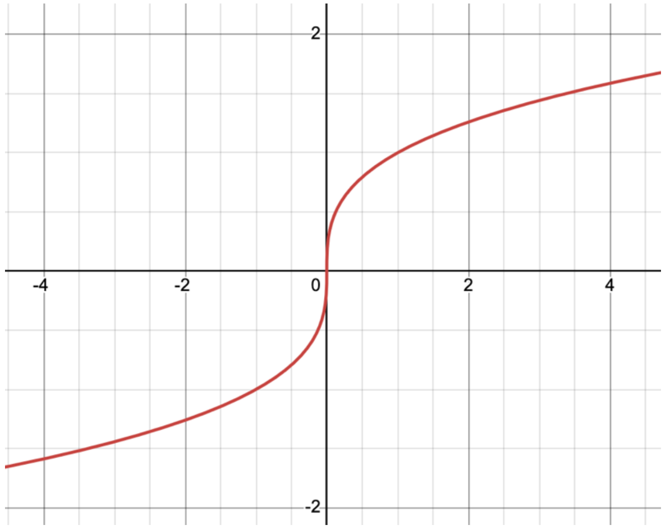
Given that $x = y^3$ find $\frac{dy}{dx}$

Note: It's important to ask yourself if the equation is a one-to-one function!

$$x = y^3$$

$$\frac{dx}{dy} = 3y^2$$

$$\therefore \frac{dy}{dx} = \frac{1}{3y^2}, \quad y \neq 0$$



Note: Whilst it would be easy to re-arrange, I would have to assume that the learning is leading into something more profound later in the course

Note: A one-to-one function is a function where each input value corresponds to a unique output value, meaning no two different inputs map to the same output.

$$y = x^{1/3}$$

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3}$$

$$= \frac{1}{3} \cdot \frac{1}{x^{2/3}}$$

Examples of when this would be used

Find the gradient of the curve $x = y^2 - 4y$ at the point where $y = 3$

Note: This is NOT a one-to-one function. However, the point we are being asked to consider lies in a section of the graph that is one to one!

$$\frac{dx}{dy} = 2y - 4$$

$$2y - 4 = 0$$

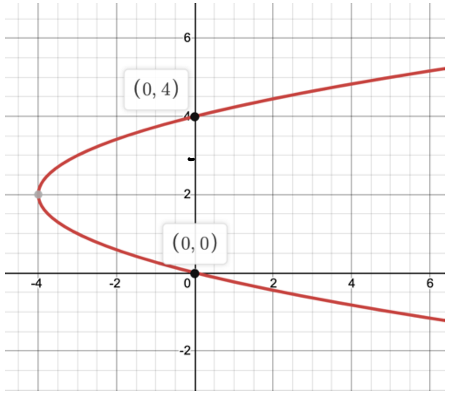
$$2y = 4$$

$$y = 2$$

$$\frac{dy}{dx} = \frac{1}{2y-4}, \quad y \neq 2$$

$$\frac{dy}{dx} = \frac{1}{6-4} = \frac{1}{2}$$

Note: A one-to-one function is a function where each input value corresponds to a unique output value, meaning no two different inputs map to the same output.



Examples of when this would be used

Find the gradient of the curve $x = y^2 - 4y$ at $x = 5$.

Note: it's important to notice that this value of x needs us to consider two one-to-one functions. One with a domain $y \geq 2$ and the other with domain $y \leq 2$

$$\frac{dx}{dy} = 2y - 4$$

$$\frac{dy}{dx} = \frac{1}{2y-4}$$

$$\therefore \frac{dy}{dx} = \frac{1}{6} \text{ @ } y = 5$$

$$\frac{dy}{dx} = \frac{-1}{6} \text{ @ } y = -1$$

$$y^2 - 4y = x$$

$$y^2 - 4y = 5$$

$$y^2 - 4y - 5 = 0$$

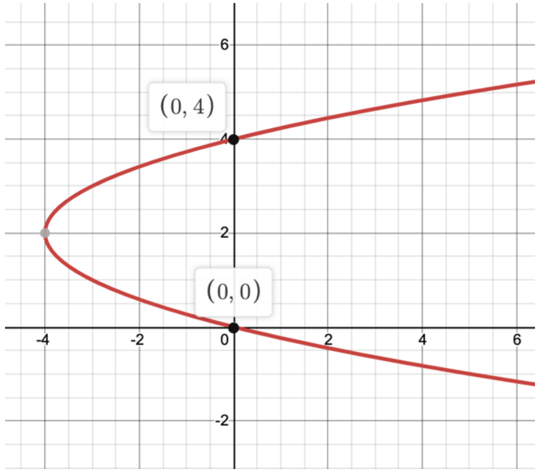
$$(y-2)^2 - 9 = 0$$

$$(y-2)^2 = 9$$

$$y - 2 = \pm 3$$

$$y = 5 \text{ or } -1$$

Note: A one-to-one function is a function where each input value corresponds to a unique output value, meaning no two different inputs map to the same output.



Using the CAS

The CAS is awesome and can do a lot of this for you!

$x = y^2 - 4y$

$\text{solve}(x=y^2-4\cdot y, y)$

$y=-(\sqrt{x+4}-2) \text{ or } y=\sqrt{x+4}+2$

$\frac{d}{dx}\left(-(\sqrt{x+4}-2)\right) \qquad \frac{-1}{2\cdot\sqrt{x+4}}$

$\frac{d}{dx}\left(\sqrt{x+4}+2\right) \qquad \frac{1}{2\cdot\sqrt{x+4}}$

$\frac{d}{dx}\left(-(\sqrt{x+4}-2)\right)|x=5 \qquad \frac{-1}{\;}$

$\frac{d}{dx}\left(\sqrt{x+4}+2\right) \qquad \frac{1}{2\cdot\sqrt{x+4}}$

$\frac{d}{dx}\left(-(\sqrt{x+4}-2)\right)|x=5 \qquad \frac{-1}{6}$

$\frac{d}{dx}\left(\sqrt{x+4}+2\right)|x=5 \qquad \frac{1}{6}$

Learning Objectives: Revisited

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand how we can find the derivatives of $x = f(y)$.

Work to be completed

The following represents the minimum work which should be completed.

The more questions you answer from each exercise, chapter review and Checkpoints the better you chance of gaining an excellent study score in November.

Specialist Mathematics Units 3 and 4 Textbook

Chapter 8
Exercise 8B: Derivatives of $x = f(y)$
Questions: All questions

Note: I have been advised to set you all questions from Chapters 6 and 7 to ensure you are sufficiently prepared for Chapters 8, 9 and 10. The exact quote I have is, "*If they don't do them all they won't be able to access the content in those chapters and may well fail*".