Differentiation

Year 12 Specialist Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Review the work on differentiation from previous years
- Recap the following important rules:
    - Product rule
    - Quotient rule
    - Chain rule
- Recall the derivatives of trigonometric functions
- Remember that we can express differentiation using different notations
- Recall the derivative of log functions.

Recap

This is the first lesson in a new section of the Specialist Mathematics Units 3 and 4 course. Happily, this is a topic we have covered before but we now take it a little deeper.

Depending on where you are with Mathematics Methods you may not have covered the Product, Chain and Quotient rules yet. That's not a major problem as I'm going to show you how and when to use them. When they are covered in Methods, make sure you listen though! They will provide more context.

Differentiation from first principals

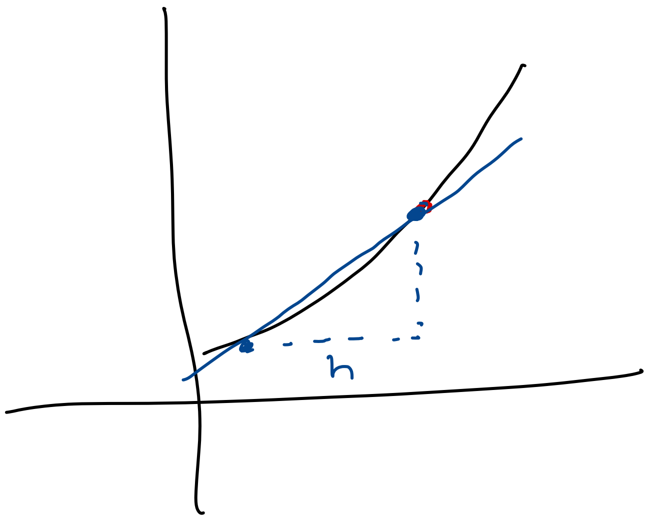
This is awesome … and just provides context for how differentiation works.

We can show that to find the gradient at a point we can use:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

It’s important to remember that when you’re differentiation your finding the equation of the tangent to a function.

Substituting an x value into this equation will give you the gradient of the tangent to the line at that point.



Gradient is also equal to the angle made with the x-axis

$$\tan\theta = \frac{O}{A}$$

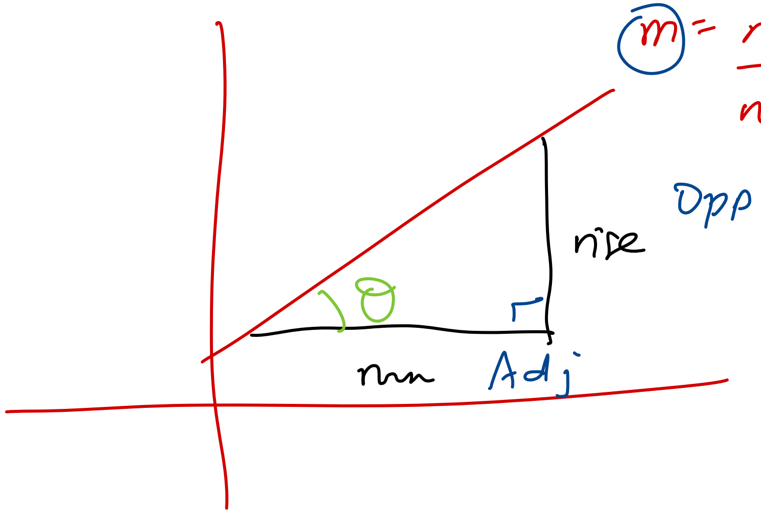
We can also show that the gradient at a point can be given by:

$$m = \tan\theta$$

Where θ is the clockwise angle between the line and the positive direction of the x-axis.

$$m = \tan\theta$$

$$m = \frac{\text{rise}}{\text{run}} = \frac{O}{A}$$



The product rule

The product rule allows us a way to find the differential when there are two terms multiplied together. This is generally used when it's not practical to multiply the expression out into individual terms or it's not possible to do so.

We use the rule that if we can write $f(x) = g(x)h(x)$ then $f'(x) = g'(x)\,h(x) + g(x)\,h'(x)$

Example: Differentiate the following with respect to x

$\sqrt{x}\sin x$

$$y' = uv' + vu'$$

$$u = \sqrt{x} \qquad v = \sin x$$

$$u = x^{1/2} \qquad v' = \cos x$$

$$u' = \frac{1}{2}x^{-1/2}$$

$$y = \underset{u}{\sqrt{x}} \cdot \underset{v}{\sin x}$$

$$\therefore y' = x^{1/2} \cdot \cos x + \sin x \cdot \frac{1}{2}x^{-1/2}$$

$$y' = \cos x\sqrt{x} + \frac{1}{2} \cdot \frac{\sin x}{\sqrt{x}}$$

The quotient rule

The quotient rule allows us a way to find the differential when there are two terms divided. This is generally used when it's not practical to divide the expression out into individual terms or it's not possible to do so.

We use the rule that if we can write $f(x) = \frac{g(x)}{h(x)}$ then $f'(x) = \frac{g'(x)h(x) - g(x)\,h'(x)}{(h(x))^2}$

$$y' = \frac{vu' - uv'}{v^2}$$

Example: Differentiate the following with respect to x

$\frac{x^2}{\sin x}$ u v

$u = x^2$ $v = \sin x$

$u' = 2x$ $v' = \cos x$

$$y' = \frac{\sin x \cdot 2x - x^2 \cdot \cos x}{\sin^2 x}$$

The chain rule

The chain rule allows us a way to find the differential when there is a function within a function and it makes sense to separate them and consider them as separate functions.

We use the rule that if we can write $f(x) = h(g(x))$ then $f'(x) = h'\big(g(x)\big)\, g'(x)$

Example: Differentiate the following with respect to x

$\cos(x^2 + 1)$

$u = x^2 + 1$

$\frac{du}{dx} = 2x$

$y = \cos(x^2 + 1)$

$y = \cos u$

$\frac{dy}{du} = -\sin u$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin(x^2+1) . 2x$

$= -2x . \underline{\underline{\sin}}\,(x^2+1)$

Differentials of trigonometric functions

We have met, in previous courses the following standard differentials:

$f(x) = \sin x$ then $f'(x) = \cos x$

$f(x) = \cos x$ then $f'(x) = -\sin x$

There is one other that may have been mentioned, which becomes more important in Specialist Mathematics and that is …

$f(x) = \tan(kx)$ then $f'(x) = k\sec^2(kx)$

y	y'
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

$$\tan x = \frac{\sin x}{\cos x} \quad \begin{matrix} u \\ v \end{matrix}$$

Examples of Differentials of trigonometric functions

Differentiate each of the following with respect to x:

$\mathbf{tan(5x^2+3)}$

$\tan^3 x$

$\sec^2(3x)$

$y = \tan(5x^2+3)$

$u = 5x^2+3$

$y = \tan u$

$\frac{du}{dx} = 10x$

$\frac{dy}{du} = \sec^2 u$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \sec^2(5x^2+3).10x$

$= 10x.\sec^2(5x^2+3)$

Examples of Differentials of trigonometric functions

Differentiate each of the following with respect to x:

$\tan(5x^2 + 3)$

$\boldsymbol{\tan^3 x}$

$\sec^2(3x)$

$y = (\tan x)^3$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$y = u^3$

$\frac{dy}{du} = 3u^2$

$\therefore \frac{dy}{dx} = 3(\tan x)^2 \cdot \sec^2 x$

$= \underline{\underline{3\tan^2 x \cdot \sec^2 x}}$

Examples of Differentials of trigonometric functions

Differentiate each of the following with respect to x:

$\tan(5x^2 + 3)$

$\tan^3 x$

$\mathbf{sec^2(3x)}$

Remember: $\sec^2 x = \tan^2 x + 1$

$\tan(kx) \Rightarrow k.\sec^2(kx)$

$\sec^2(3x) = \tan^2(3x) + 1$ ✓

$= (\tan(3x))^2 + 1$

$\frac{dy}{dx} = [2\tan(3x)]' . 3\sec^2(3x)$

$= \underline{\underline{6\tan(3x).\sec^2(3x)}}$

Operator notation

Maths is nothing more than a Big Fat Trick. It's designed to confuse you using smoke and mirrors.

We have a number of ways of asking for a differentiation to take place:

$$f'(x) = \frac{dy}{dx}$$

There is also operator notation:

$$\frac{d}{d(x)}\left(f(x)\right)$$

$f'(x)$

$\frac{dy}{dx}$

$\frac{d(f(x))}{dx}$

Examples of operator notation

Find:

$\frac{d}{dx}(x^2 + 2x + 3)$ $= 2x + 2$

$\frac{d}{dx}(e^{x^2})$ $= 2x . e^{x^2}$

$\frac{d}{dz}(\sin^2(z))$

$y = e^{x^2}$ $u = x^2$

$y = e^u$ $\frac{du}{dx} = 2x.$

$\frac{dy}{du} = e^u$

$\therefore \frac{dy}{dx} = e^{x^2} . 2x = \underline{\underline{2x . e^{x^2}}}$

$y = (\sin x)^2$

$y' = 2 \sin x . \cos x$

$= \underline{\underline{\sin 2x}}$

The derivative of $\log_e |x|$

$$|x| = \begin{cases} x, & x > 0 \\ -x & x < 0 \end{cases}$$

This function is about to become very important in this course (and the Methods 3 and 4 course)

Let's look at how to find the derivative …

Find $\frac{d}{dx}(\log_e|x|)\ for\ x \neq 0$

$y = \log_e |x|$

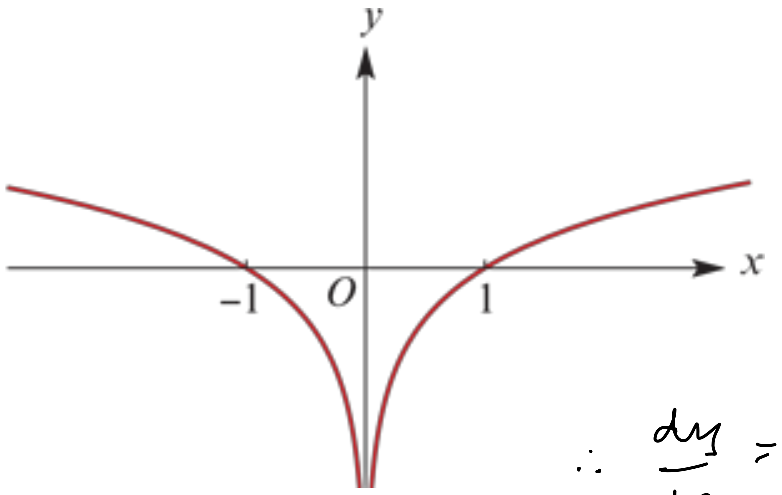
$y = \log_e x \quad , x > 0$

$\frac{dy}{dx} = \frac{1}{x}$ ✓

$y = \log_e (-x) \quad x < 0$

$y = \log_e u$

$\frac{dy}{du} = \frac{1}{u}$



$u = -x$

$\frac{du}{dx} = -1$

$\therefore \frac{dy}{dx} = \frac{1}{-x} \cdot -1$

$= \frac{1}{x}$

Using the result of the derivative of $\log_e |x|$

We can now use the result in the following way

Find $\frac{d}{dx}(\log_e|\sec x|)$ $for\ x \notin \left\{\frac{(2k+1)\pi}{2} : k \in \mathbb{Z}\right\}$

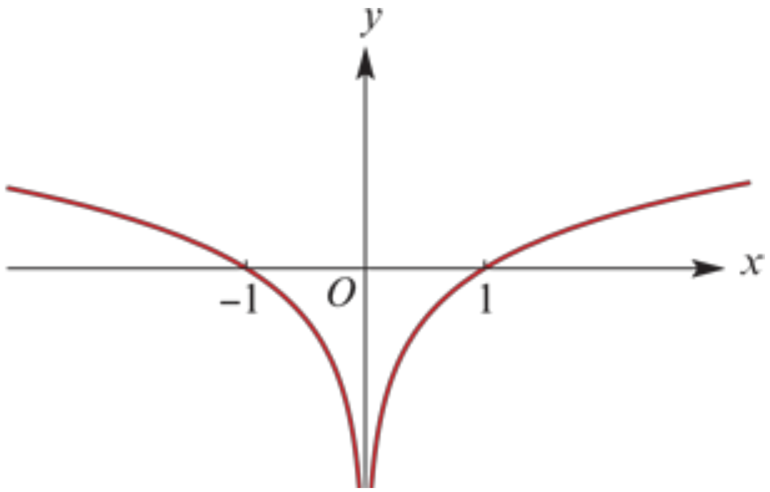
$y = \log_e |\sec x|$

$y = \log_e \left|\frac{1}{\cos x}\right|$

$y = \log_e \frac{|1|}{|\cos x|}$

$y = \log_e \frac{1}{|\cos x|}$

$y = \log_e (|\cos x|)^{-1}$



$\therefore y = -1 . \log_e |\cos x|$

$= +1 . \frac{1}{\cos x} . +\sin x$

$= \tan x$

Work to be completed

The following represents the minimum work which should be completed.

The more questions you answer from each exercise, chapter review and Checkpoints the better you chance of gaining an excellent study score in November.

Specialist Mathematics Units 3 and 4 Textbook

Chapter 8
Exercise 8A: Differentiation
Questions: All questions

Note: I have been advised to set you all questions from Chapters 6 and 7 to ensure you are sufficiently prepared for Chapters 8, 9 and 10. The exact quote I have is, "*If they don't do them all they won't be able to access the content in those chapters and may well fail*".