Translations of functions

Year 11

Mathematical Methods

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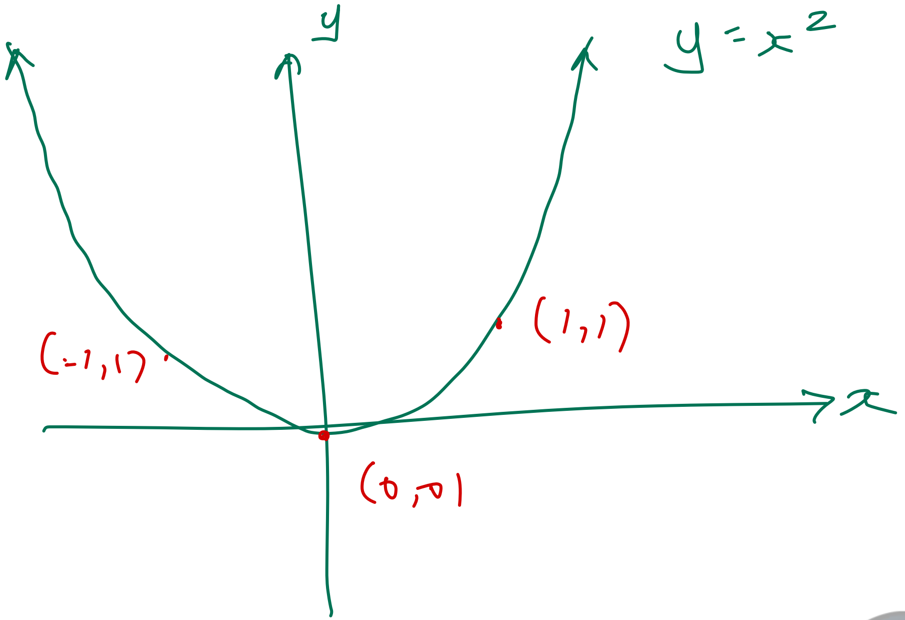
Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Year 11 Mathematical Methods course.

- Know what it means to translate a function
- Know that it means by the term “Cartesian Plane”
- Understand the notation of translation of functions
- Apply translations to sketch graphs



RECAP

This is the start of a new section of the Mathematical Methods course and is going to look at transformations of functions. This is a pretty massive part of the course and will build on the work which has been covered previously.

It's going to be important to understand the "basic" shape of the more common functions so we can sketch their "base graph" which will provide the basis for translating them.

$y = \frac{1}{x^2}$

$y = x^2$

$y = x^3$

$y = \sqrt{x}$

$x^2 + y^2 = r^2$

$y = \pm\sqrt{r^2 - x^2}$

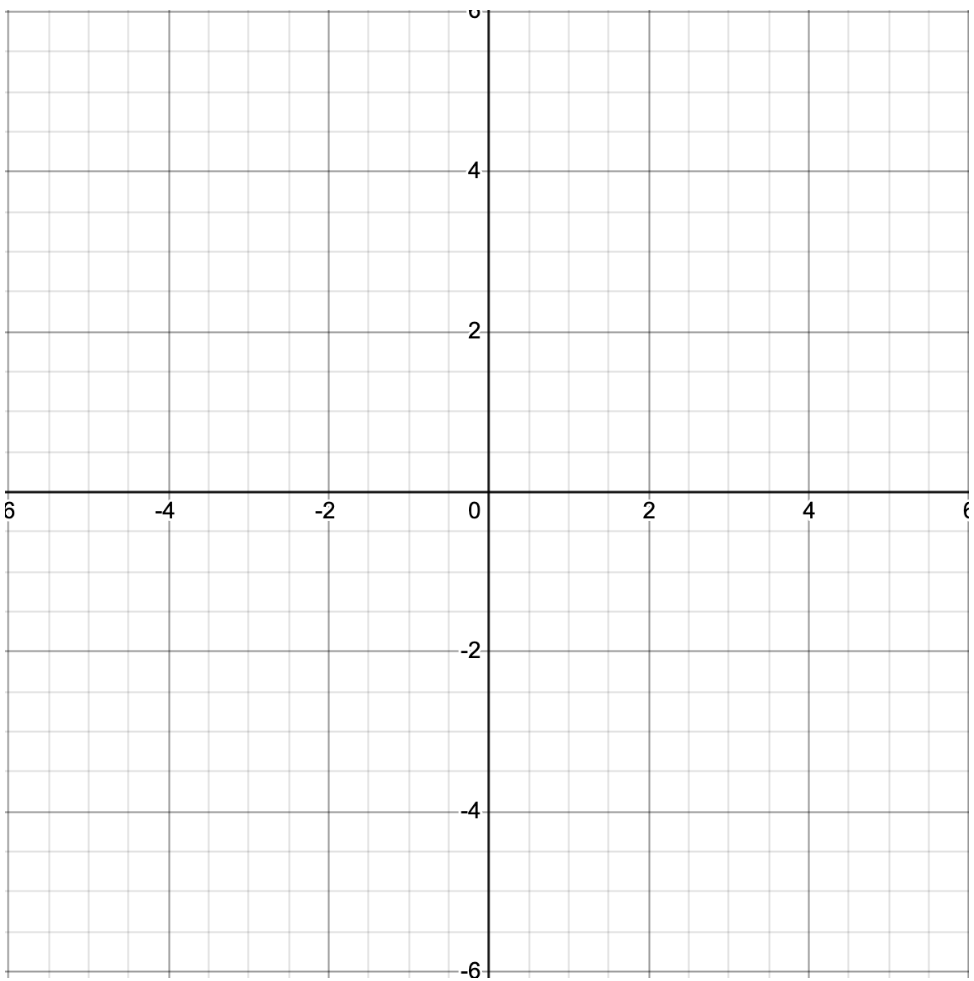
$y = \frac{1}{x}$

What is the Cartesian Plane?

I like to think of this as a fancy name for a set of axes.

To quote the Cambridge Textbook, the Cartesian plane is represented by the set $\mathbb{R}^2$ of all ordered pairs of real numbers”. Does that make it any clearer?

Transformations in this section of the course associate each ordered pair of $\mathbb{R}^2$ with a unique ordered pair.



Notation

Every point on a graph has a co-ordinate expressed in the following form:

$$(x, y)$$

That is, there is an x- value and an y-value.

We can do a range of things to this co-ordinate. For example, we could move it. We could move it up, down, left or right. To do so would be to **translate it**.

- When we move things up and down we would be changing only the y-coordinate.
- When we move the coordinate left or right, we would be changing the x-coordinate

If we do this to every point on the graph we could express this translation using algrebra:

$$(x + 3, y)$$

This would mean that every x coordinate has had 3 added to it. This would have the effect of moving the graph 3 units to the right.

Notice that the y-coordinates will not change.

$\rightarrow 2$

$(1, 2)$

$(3, 2)$

Notation

We can, of course change just the y-coordinate.

$$(x, y-2)$$

This would have the effect of moving the whole graph down by 2 units. The graph would remain the same horizontally as we haven't changed the x-coordinates.

Or, we can change both!

$$(x-1, y+4)$$

This would move the whole graph one unit to the left, and four units up.

We can think of each point as now having an image.

From previous years, we know that we can express this using the following notation:

$$(x', y')$$

$(x, y) \longrightarrow (x', y')$

Notice the dashed notation.

Why is this important?

Isn't that the big question for the whole of Mathematics?

Let's look at an example …

Imagine we have the graph of $y = x^2$ and I want to move the whole graph 2 units to the right and 4 units up.

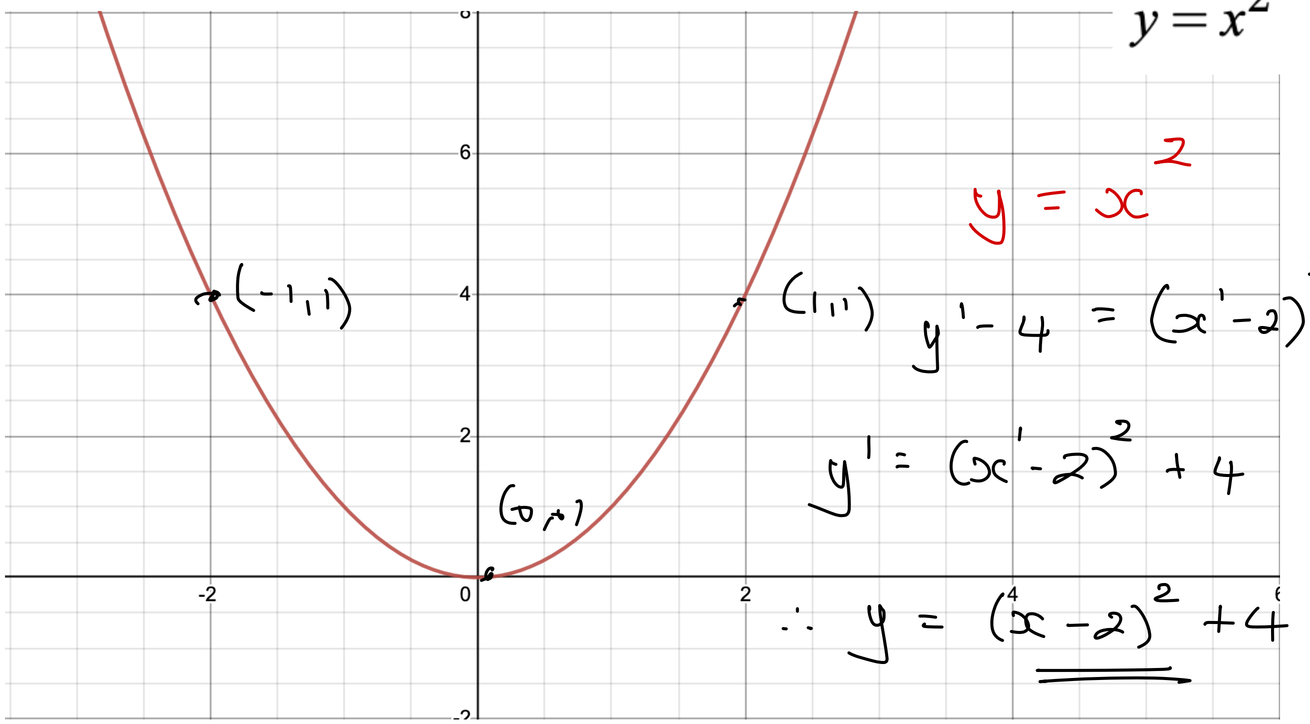
We can sketch this quite easily … but what would be the equation of the new curve (its image)?

We can use some pretty funky (and easy) algebra to help us find out.

$$(x, y) \rightarrow (x', y')$$

$$\downarrow$$

$$(\underbrace{x+2}_{x'}, \underbrace{y+4}_{y'})$$



$y = x^2$

$y' - 4 = (x' - 2)^2$

$y' = (x' - 2)^2 + 4$

$\therefore y = (x - 2)^2 + 4$

$\therefore x' = x + 2 \qquad y' = y + 4$

$x = x' - 2 \qquad y = y' - 4$

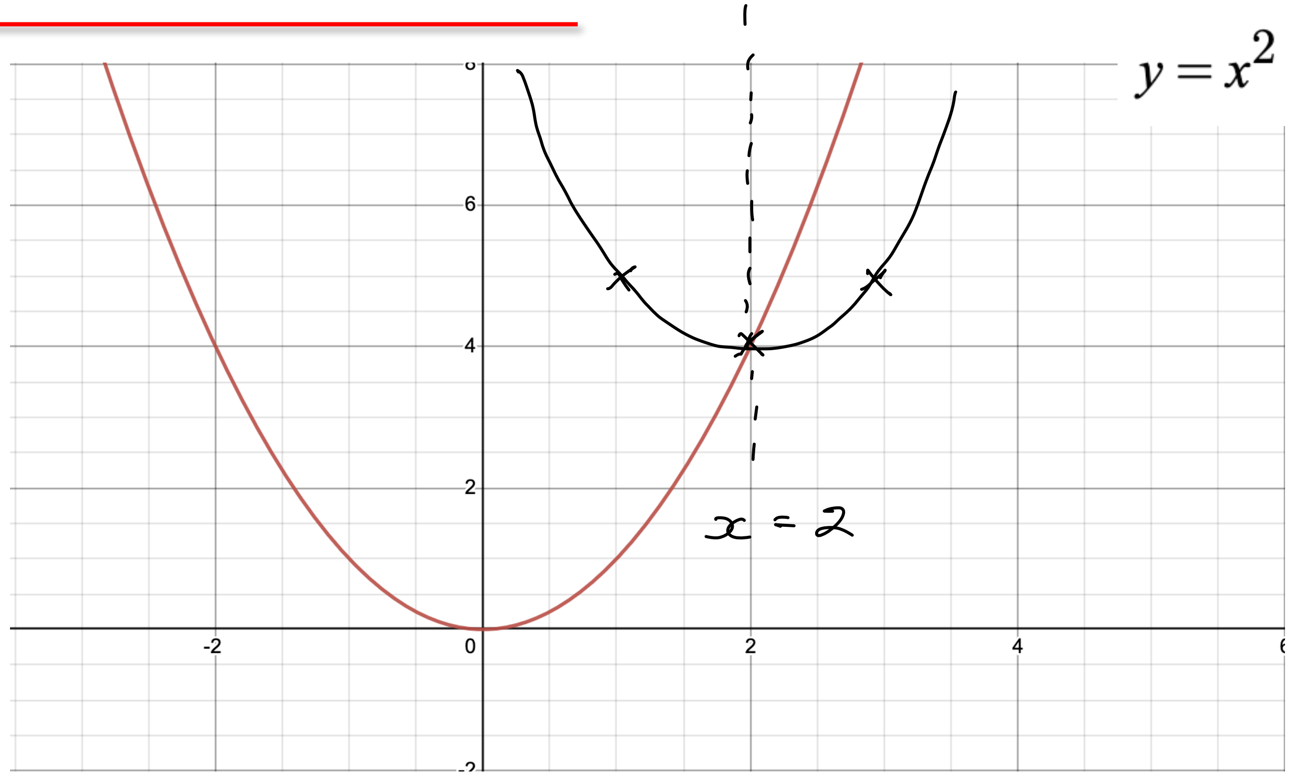
Why is this important?

So, the translated function (or image) will have the following equation:

$$y = (x - 2)^2 + 4$$

Which happens to be completed square form!

We've met this before and, probably could have gone this way straight away … but we're learning the basics to apply to more complex problems later.



Another way of doing it

There are, of course, other ways of doing it.

I think of this method as a “short cut” and should be used with caution as the order you apply transformations is important. More on this another time.

Here is a helpful table.

Let’s apply it to the example we had before taking the graph of $y = x^2$ two units to the right and 4 units up.

Positive x translation by h units Replace every instance of $\mathrm{x} \;{-}{>}\; \mathrm{x} - \mathrm{h}$	**Negative x translation by h units** Replace every instance of $\mathrm{x} \;{-}{>}\; \mathrm{x} + \mathrm{h}$
Positive y translation by k units Replace every instance of $\mathrm{y} \;{-}{>}\; \mathrm{y} - \mathrm{k}$	**Negative y translation by k units** Replace every instance of $\mathrm{y} \;{-}{>}\; \mathrm{y} + \mathrm{k}$

$$y = x^2$$

$$y = (x-2)^2$$

$$y - 4 = (x-2)^2$$

$$\underline{\underline{y = (x-2)^2 + 4}}$$

$\rightarrow 2$ ✓ $\uparrow 4$ ✓

Example 1

Find the equation for the image of the curve with equation $y = f(x)$, where $f(x) = \dfrac{1}{x}$, under a translation 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

$\rightarrow 3 \quad x \to x - 3$

$\downarrow 2 \quad y \to y + 2$

$$(x, y) \to (\underset{x'}{x+3}, \underset{y'}{y-2})$$

$\therefore x' = x + 3 \qquad y' = y - 2$

$x = x' - 3 \qquad y = y' + 2$

$$y = \frac{1}{x}$$

$$y' + 2 = \frac{1}{x' - 3}$$

$$y = \frac{1}{x - 3} - 2$$

$$y = \frac{1}{x}$$

$$y = \frac{1}{x - 3}$$

$$y + 2 = \frac{1}{x - 3}$$

$$y = \frac{1}{x - 3} - 2$$

Applying transformations in reverse

Knowing this theory means we can now sketch some pretty complex graphs.

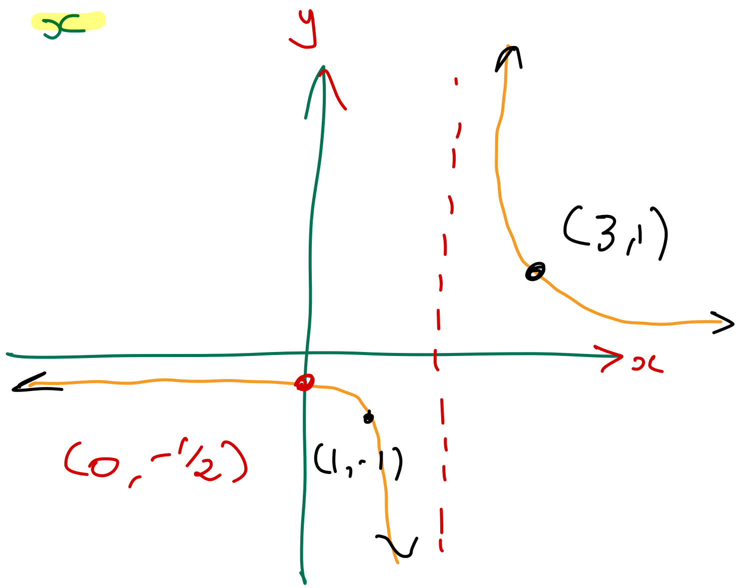
For example, how would we sketch the following graph?

$$y = \frac{1}{x-2}$$

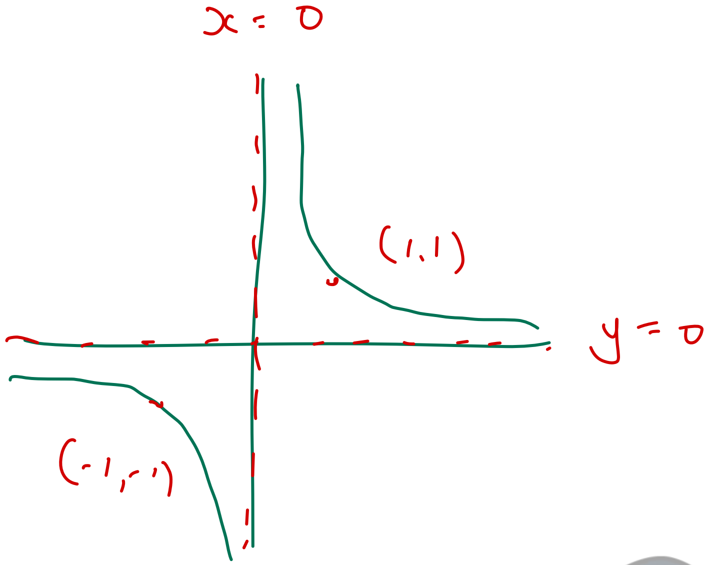
$x = 0$

$y = -\frac{1}{2}$

$y = \frac{1}{x}$



Think about:
What could the "base" function have been before it was transformed?
What could the "original x" have been replaced with to get this new function?

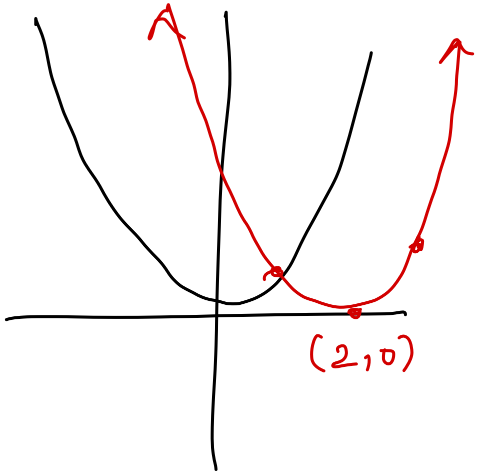


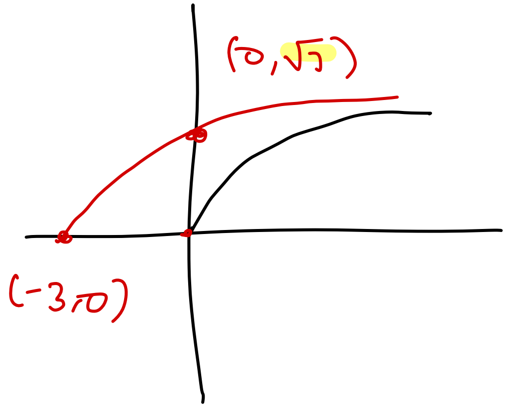
Applying transformations in reverse

What about the following two functions?

$$y = (x - 2)^2 \qquad y = x^2$$

$$y = \sqrt{x + 3} \qquad y = \sqrt{x}$$





Think about:
What could the "base" function have been before it was transformed?
What could the "original x" have been replaced with to get this new function?

Example: Sketching graphs

It is common, in exams and exercises to be given a “base” function and then asked to sketch a transformed version of the function.

For example, for $y = f(x) = \frac{1}{x}$, sketch the graph of each of the following, labelling asymptotes and axis intercepts.

a) $y = f(x+3)$
b) $y = f(x) - 3$

$y = f(x)$

$y = f(x+3)$

$y = \frac{1}{x}$

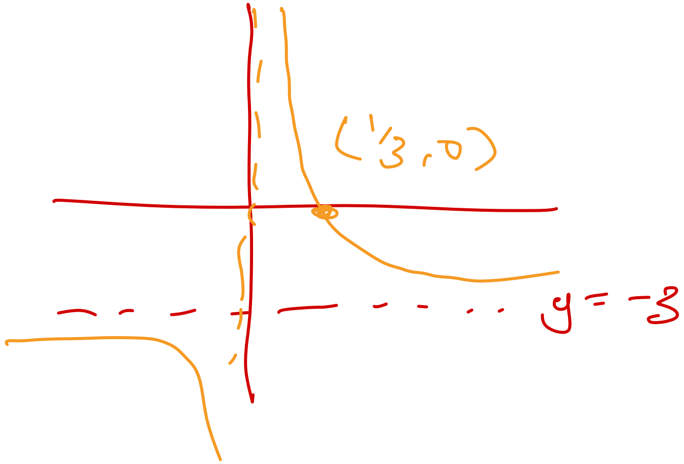
$y = f(x+3) = \frac{1}{x+3}$

$y = f(x) - 3$

$= \frac{1}{x} - 3$

$0 = \frac{1}{x} \quad 3$

$3 = \frac{1}{x} \quad x = \frac{1}{3}$

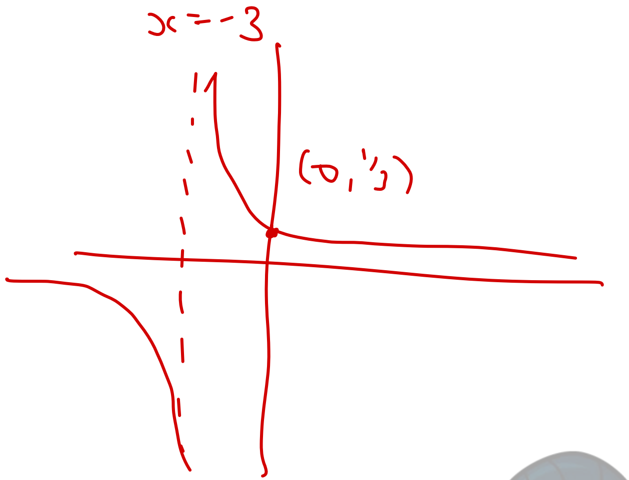


Important information

It is VITAL, when sketching graphs that you always include the following:

- Axis intercepts
- Asymptotes
- Turning points
- End points (if domain is limited)

Always make sure you draw them in pencil.



Example 2: Sketching graphs

For $y = f(x) = x^2$, sketch the graph of the following, labelling asymptotes and axis intercepts.

$y + 2 = f(x + 1)$

$$y + 2 = (x+1)^2$$

$$y = (x+1)^2 - 2$$

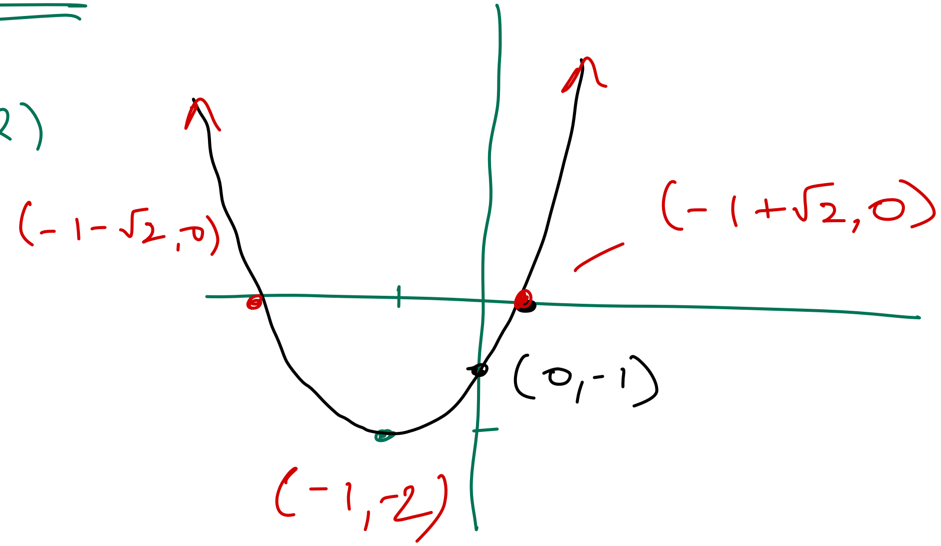
$(-1, -2)$

$$0 = (x+1)^2 - 2$$

$$2 = (x+1)^2$$

$$\pm\sqrt{2} = x + 1$$

$$x = -1 \pm \sqrt{2}$$



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