

Solving quadratic equations over the complex numbers

Year 12 Specialist Maths
Units 3 and 4

www.maffsguru.com

Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand how to factorise quadratics over the complex plane
- Solve quadratic equations over the complex plane
- Understand how to use the CAS to assist with the above



Recap

We have so far looked at a number of things which relate to complex numbers:

- What a complex number is and how it is represented
- How to find the length (modulus) and conjugate of a complex number
- How to divide complex numbers
- How to express complex numbers in the modulus-argument form (polar form)
- How to perform basic operations on complex numbers written in polar form

Now it's time to look at how we can use complex numbers to solve quadratics! This really is Maths for the fun of doing Maths!!!



RECAP: Factorisation … what is it?

We remember, from the rectangular plane, that when we solve a quadratic equation, we are trying to find solutions which are primarily roots of the equation. This is where the equation crosses the x-axis.

Example:

$$y = x^2 + x - 2$$

Can be rewritten as:

$$x^2 + x - 2 = y$$

When y=0 it crosses the x-axis

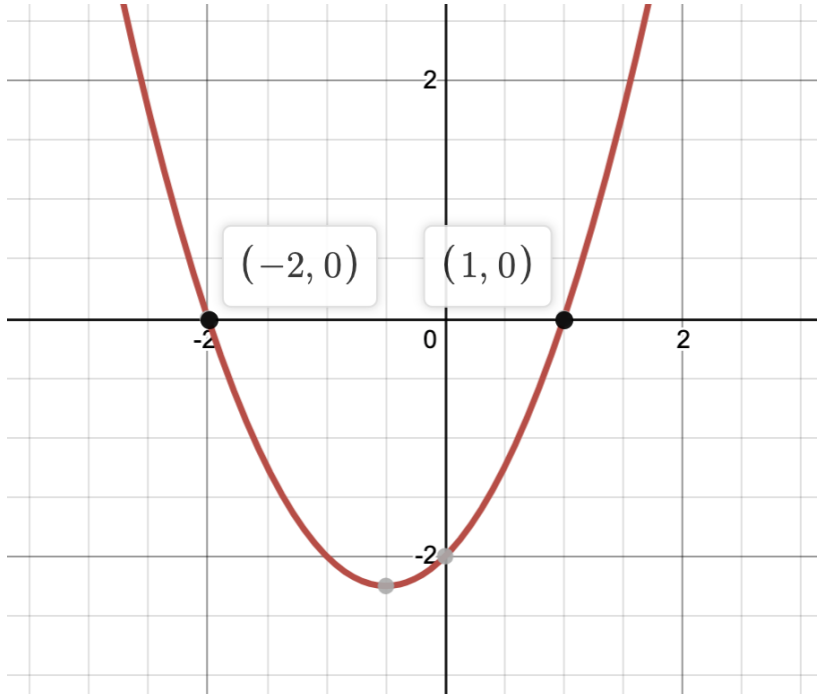
$$x^2 + x - 2 = 0$$

This factorises to:

$$(x + 2)(x - 1) = 0$$

And hence has solutions at:

$$x = 1 \; and \; x = -2$$



Here was can see there are two **real** solutions.

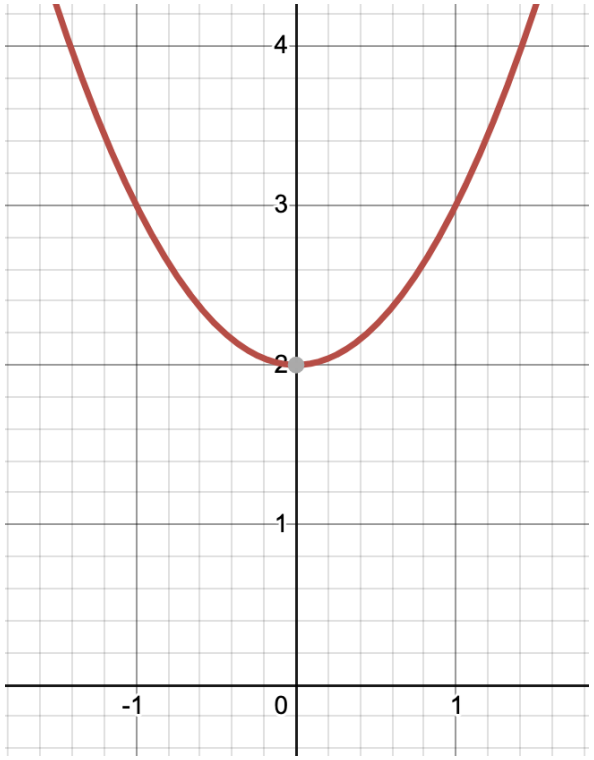


RECAP: Factorisation … what is it?

When we look at the function $y = x^2 + 2$ we can see that there are no **real solutions** but there are going to be two complex solutions!

So … the learning from this lesson is that there will always be (a maximum) of two solution when we solve quadratics.

$$\sqrt{b^2 - 4ac}$$



Here was can see there are no **real** solutions.



Why do we need complex numbers?

Remember, we cannot solve quadratics with a negative discriminant:

$$\sqrt{b^2 - 4ac} \not< 0$$

This generally means that the graph sits above the x-axis and hence doesn't cross or have any roots.

Now that we have a way of looking at the square root of a negative number, we can now solve quadratics!

We know that $i^2 = -1$

So, let's start with a quadratic with the form $z^2 + a^2$

$$z^2 + a^2 = z^2 - a^2 i^2 = z^2 - (ai)^2 = (z + ai)(z - ai)$$

$i^2 = -1$

$z^2 + a^2$

$= z^2 - a^2 i^2$

$= z^2 - (ai)^2$

$\therefore (z - ai)(z + ai)$



Examples

Factorise the following:

$\boldsymbol{z^2 + 16}$

$2z^2 + 6$

$$z^2 + 16 = z^2 - 16i^2$$

$$= z^2 - (4i)^2$$

$$\therefore z^2 + 16 = (z + 4i)(z - 4i)$$



Examples

Factorise the following:

$z^2 + 16$

$\mathbf{2z^2 + 6}$

$$2(z^2+3) = 2[z^2 - 3i^2]$$

$$= 2[z^2 - (\sqrt{3}i)^2]$$

$$= 2(z - i\sqrt{3})(z + i\sqrt{3})$$



More examples

Factorise the following:

$z^2 + z + 3$

$2z^2 - z + 1$

$2z^2 - 2(3 - i)z + 4 - 3i$

$2^{3/4}$

$$\left(z + \frac{1}{2}\right)^2 - \frac{1}{4} + 3$$

$$= \left(z + \frac{1}{2}\right)^2 - \frac{11i^2}{4}$$

$$= \left(z + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{11}i}{2}\right)^2$$

$$\therefore \left(z + \frac{1}{2} + \frac{\sqrt{11}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{11}}{2}i\right)$$

$=$



More examples

Factorise the following:

$z^2 + z + 3$

$\mathbf{2z^2 - z + 1}$

$2z^2 - 2(3 - i)z + 4 - 3i$

$$2\left[z^2 - \frac{1}{2}z + \frac{1}{2}\right] = 2\left[\left(z - \frac{1}{4}\right)^2 - \frac{1}{16} + \frac{1}{2}\right]$$

$$= 2\left[\left(z - \frac{1}{4}\right)^2 + \frac{7}{16}\right]$$

$$= 2\left[\left(z - \frac{1}{4}\right)^2 - \left(\frac{\sqrt{7}i}{4}\right)^2\right]$$

$$\therefore 2\left(z - \frac{1}{4} + \frac{\sqrt{7}i}{4}\right)\left(z - \frac{1}{4} - \frac{\sqrt{7}i}{4}\right)$$

$=$



More examples

Factorise the following:

$z^2 + z + 3$

$2z^2 - z + 1$

$\mathbf{2z^2 - 2(3 - i)z + 4 - 3i}$

$$2\left[z^2 - (3-i)z + 2 - \frac{3i}{2}\right]$$

$$= 2\left[\left(z - \frac{(3-i)}{2}\right)^2 - \frac{(3-i)^2}{4} + \frac{4 - 3i}{4}\right]$$

$$= 2\left[\left(z - \frac{(3-i)}{2}\right)^2 - \frac{8-6i}{4} + \frac{8-6i}{4}\right]$$

$$= 2\left(z - \frac{(3-i)}{2}\right)^2$$

$$(3-i)^2 \qquad -1$$
$$= 9 - 6i + i^2$$
$$= 8 - 6i$$



Using the quadratic formula

This was one of the first things I was made to learn when I was 11 years old and joining my new maths class in my new school. It's no wonder I was terrified of Mathematics!

$$x = \frac{\left(-b \pm \sqrt{b^2 - 4ac}\right)}{2a}$$

Using the same idea but with $az^2 + bz + c = 0$

Then the formula (giving complex solutions) becomes:

$$z = \frac{\left(-b \pm \sqrt{b^2 - 4ac}\right)}{2a}$$



Examples

Solve each of the following equations for z

$\boldsymbol{z^2 + z + 3 = 0}$

$2z^2 - z + 1 = 0$

$z^2 = 2z - 5$

$2z^2 - 2(3 - i) + 4 - 3i = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 3}}{2}$$

$$z = \frac{-1 \pm \sqrt{-11}}{2} = \frac{-1 \pm i\sqrt{11}}{2}$$

$$z = \frac{-1 + i\sqrt{11}}{2} \quad \text{or} \quad z = \frac{-1 - i\sqrt{11}}{2}$$



Examples

Solve each of the following equations for z

$z^2 + z + 3 = 0$

$\mathbf{2z^2 - z + 1 = 0}$

$z^2 = 2z - 5$

$2z^2 - 2(3 - i) + 4 - 3i = 0$

$z^2 - 2z + 5 = 0$ $\therefore$



Examples

Solve each of the following equations for z

$z^2 + z + 3 = 0$

$2z^2 - z + 1 = 0$

$\boldsymbol{z^2 = 2z - 5}$

$2z^2 - 2(3 - i) + 4 - 3i = 0$



Examples

Solve each of the following equations for z

$z^2 + z + 3 = 0$

$2z^2 - z + 1 = 0$

$z^2 = 2z - 5$

$\mathbf{2z^2 - 2(3 - i) + 4 - 3i = 0}$

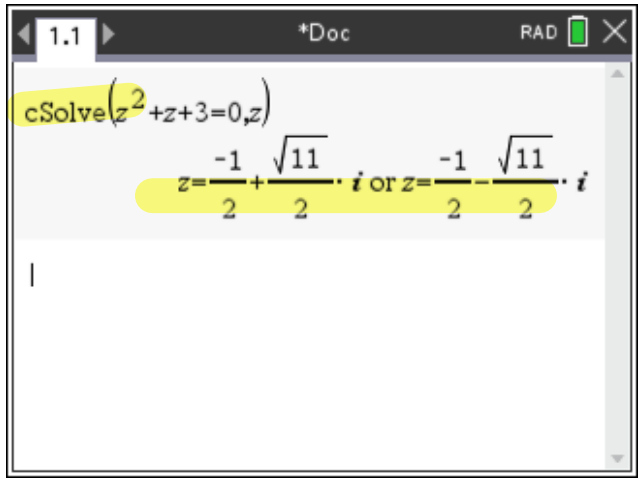


Using the CAS

We can use the CAS to help us (when we have a CAS enabled paper!)

We now have a new function called **cSolve** which means Complex Solve!

Notice: the negative sign is the difference between the two solutions. This is going to be important in later sections of the course!





VCAA Questions

Specialist Mathematics
2017 Paper 2

Question 3

The number of distinct roots of the equation $(z^4 - 1)(z^2 + 3iz - 2) = 0$, where $z \in C$, is

A. 2
B. 3
C. 4
D. 5
E. 6

Note: This question is a little before the theory of this section but is possible!



VCAA Questions

Specialist Mathematics
2013 Paper 2

Question 8

The principal arguments of the solutions to the equation $z^2 = 1 + i$ are

A. $\frac{\pi}{8}$ and $\frac{9\pi}{8}$

B. $-\frac{\pi}{8}$ and $\frac{7\pi}{8}$

C. $-\frac{7\pi}{8}$ and $\frac{\pi}{8}$

D. $\frac{7\pi}{8}$ and $\frac{15\pi}{8}$

E. $-\frac{3\pi}{4}$ and $\frac{\pi}{4}$

$z = r\,\mathrm{cis}\,\theta$

$z^2 = r^2\,\mathrm{cis}\,2\theta$

$r^2\,\mathrm{cis}\,2\theta = \sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4}\right)$

$\therefore r^2 = \sqrt{2}$

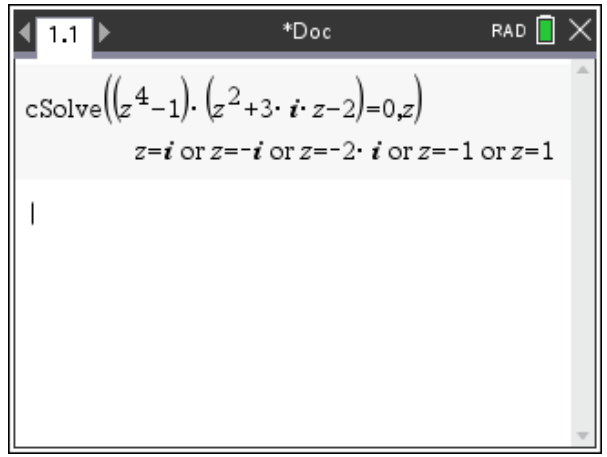
$r^2 = 2^{1/2}$

$r = 2^{1/4}$

$2\theta = \frac{\pi}{4} + 2k\pi$

$\theta = \frac{\pi}{8} + k\pi$

$\theta_1 = \frac{\pi}{8}, -\frac{7\pi}{8}$



$\frac{\pi}{8} + \pi = \frac{9\pi}{8} - \frac{16\pi}{8}$



Learning Objectives: Revisited

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand how to factorise quadratics over the complex plane
- Solve quadratic equations over the complex plane
- Understand how to use the CAS to assist with the above



Work to be completed

The following represents the minimum work which should be completed.

The more questions you answer from each exercise, chapter review and Checkpoints the better you chance of gaining an excellent study score in November.

Specialist Mathematics Units 3 and 4 Textbook

Chapter 6
Exercise 6E: Solving quadratic equations over the complex numbers
Questions: 1beh, 2bdfh, 3

