Polar form of a complex number

Year 12 Specialist Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what it means by Polar Form.
- Understand what it means by the Non-uniqueness of polar form
- Know how to find the complex conjugate in polar form
- Know how to use the CAS to solve questions

Recap

We continue our journey through the complex numbers world.

We now know how to add, subtract multiply by a scalar and divide complex numbers.
We understand what we use the conjugate for and how to represent complex numbers on an Argand Diagram,

Remember, when we draw the Argand Diagram we are representing the complex number in a cartesian form.

We're now going to meet the Polar Form which is going to blow your mind!

Barry is at it again!

I'm going to give Barry the biggest slap when I find him.

When we express complex numbers using a modulus argument form, we are also expressing it in Polar Form.

And this has nothing to do with Polar Bears!

Polar Form … making it real

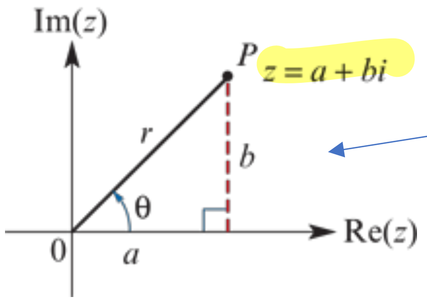
We know that we can represent a point P on an Argand Diagram using the form $z = a + bi$

Where $Re(z) = a$ and $Im(z) = b$

Thinking of this in terms of a circle, we now know that the distance from the origin to the point P can be expressed using a letter. In this case, we're going to call it r which relates to the radius from a centre point.

$a = r\cos\theta$

$b = r\sin\theta$



Using this we can see that a and b can be expressed in terms of a trigonometric ratio

$z = r\cos\theta + (r\sin\theta)i$

The polar form of a complex number

The polar form hence becomes:

$$z = r\cos\theta + (r\sin\theta)i$$

$z = r.\ \text{cis}\ \theta$

To make it easier on the eye this has a "short cut" notation of:

$$z = r\text{cis}\theta$$

The size of the angle is called the argument of z. It's normally denoted by arg z and measured anticlockwise from the horizontal axis

I remember that cos comes first and then sine due to the order of the letters in cis!

$\arg z$

$\text{Arg}\ z$

The polar form of a complex number: Using the CAS

Your CAS doesn't have a nice 'cis' function.

Hence, we can use the fact that we can enter this into the CAS using:

$$r\text{cis}\theta = re^{i\theta}$$

This can make things quicker and easier at exam time!

$$r\text{cis}\theta = 2\text{cis}\left(\frac{\pi}{3}\right)$$

$$= \underline{\underline{2e^{\frac{\pi}{3}i}}}$$

But you said modulus and argument form.

We know that r is the **length of P from the origin** which happens to be $|z|$

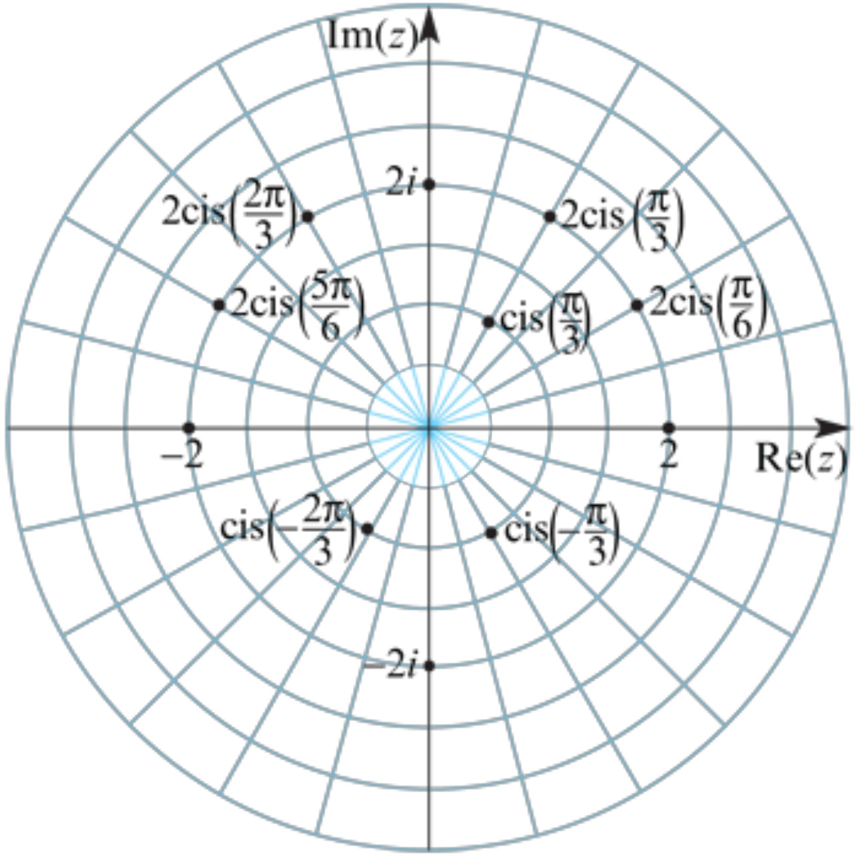
Hence, r is also the modulus of z

The angle which the complex number makes when measured **anticlockwise** from the horizontal axis is called the argument.

This seems to fit nicely with the unit circle which is also measured with respect to the horizontal axis in an anticlockwise direction.

$r = |z|$

$\frac{\pi}{12}$



More than one representation!!!

In the same way that the values of sine and cosine have repeating values every 2π the same is true of polar form!

If we're looking for **all solutions,** then we would use:

$$\cos\theta = \cos(\theta + 2n\pi)$$

$$\sin\theta = \sin(\theta + 2n\pi)$$

Arg (z)

Hence, we can write:

$-\pi < \theta \leq \pi$

$$z = rcis\theta = rcis(\theta + 2n\pi)$$

Note that the convention is to have the value of θ between $-\pi < \theta \leq \pi$. This is called the **principal value of the argument** and is denoted by $Arg\ z$ – note the capital A rather than a small a!

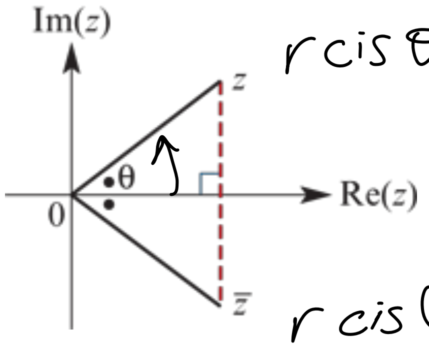
What about the complex conjugate in polar form

I knew you were wondering about this …

It makes sense to think that the complex conjugate is nothing more than a reflection of z in the horizontal axis.

Hence, we would be able to state that if $z = rcis\theta$ then $\bar{z} = rcis(-\theta)$

This is used **over and over again** in exams



$z = r cis\,\theta$

$\bar{z} = r cis(-\theta)$

Examples

Find the modulus and principal argument of each of the following complex numbers:

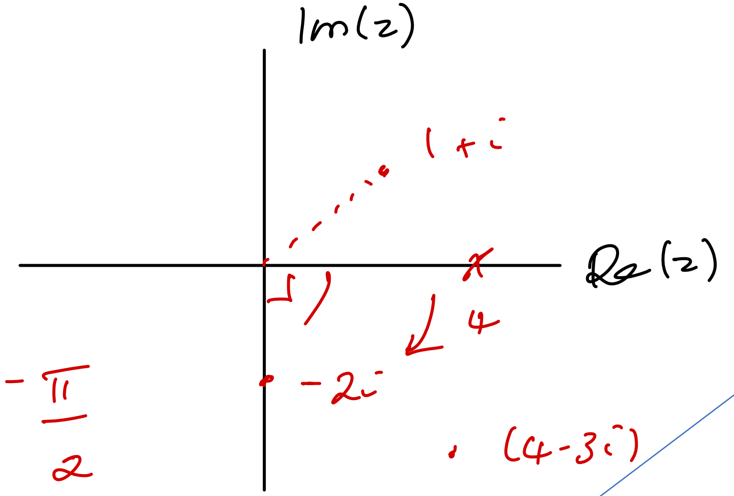
- 4
- $-2i$
- $1+i$
- $4-3i$

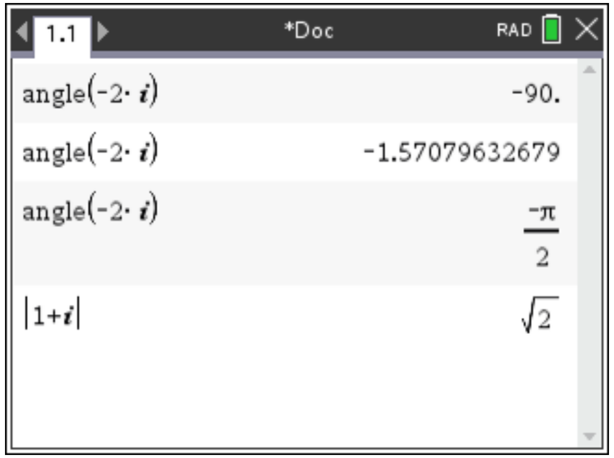
- $|4| = 4$

$\text{Arg}(4) = 0$

- $|-2i| = 2$

$\text{Arg}(-2i) = -\dfrac{\pi}{2}$





$-\pi < \theta \leq \pi$

- $|1+i| = \sqrt{2}$

$\text{Arg}(1+i) = \dfrac{\pi}{4}$

Knowing how to do this on the CAS makes life quick when doing the VCE exam

- $4-3i$ $\quad |4-3i| = \sqrt{16+9} = 5$

$\text{Arg}(4-3i) = \dfrac{-3}{4} = -\tan^{-1}\left(\dfrac{3}{4}\right)$

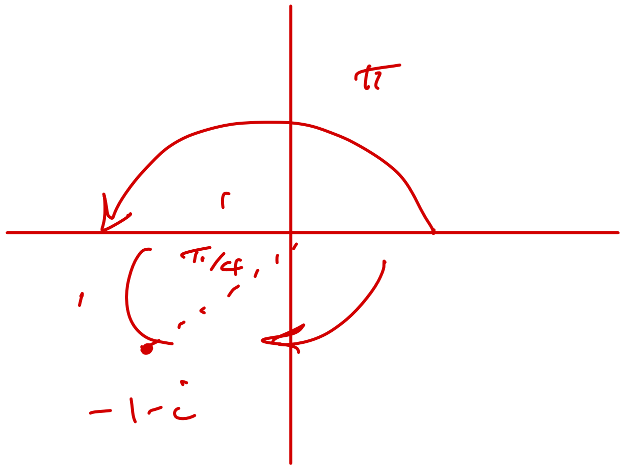
$= -0.64 \text{ rad}$

Examples

Find the **argument** of $-1-i$ in the interval $[0,2\pi]$.

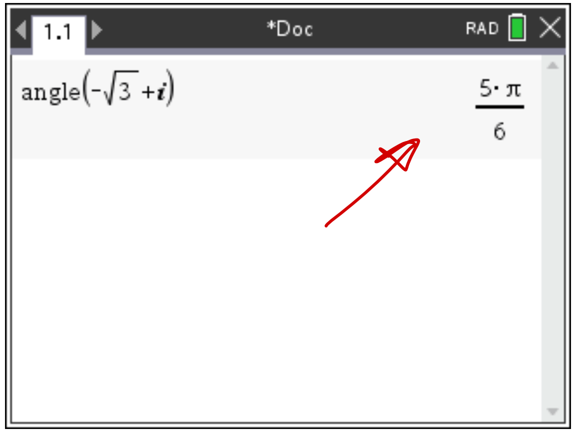
$$\arg(-1-i) = \frac{5\pi}{4}$$

$$\text{Arg}(-1-i) = -\frac{3\pi}{4}$$



Examples

Express $-\sqrt{3}+i$ in the form $r\mathrm{cis}\theta$, where $\theta = Arg(-\sqrt{3}+i)$



$$r\,\mathrm{cis}\,\theta = r\cos\theta + (r\sin\theta)\,i$$

$$= 2\,\mathrm{cis}\left(\frac{5\pi}{6}\right)$$

$$|-\sqrt{3}+i| = \sqrt{3+1}$$

$$= 2$$

$$r = 2$$

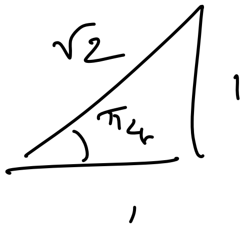
Examples

Express $2cis\left(-\frac{3\pi}{4}\right)$ in the form $a + bi$

$a + bi$

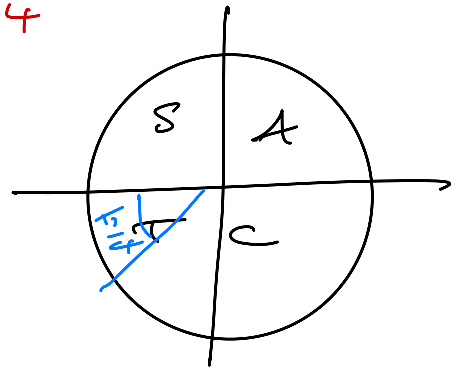
$r = 2$

$\theta = -\frac{3\pi}{4}$



$r\,cis\theta = r\cos\theta + (r\sin\theta)i$

$= 2\cos\left(-\frac{3\pi}{4}\right) + 2\sin\left(-\frac{3\pi}{4}\right)i$



$= -2\cdot\frac{1}{\sqrt{2}} - 2\cdot\frac{1}{\sqrt{2}}i$

$= -\frac{\cancel{2}\sqrt{2}}{\cancel{2}} - \frac{\cancel{2}\sqrt{2}}{\cancel{2}}i$

$= -\sqrt{2} - \sqrt{2}i$

VCAA question

VCAA 2016
Paper 2
Question 5

Question 5

If $\text{Arg}(-1+ai) = -\frac{2\pi}{3}$, then the real number a is

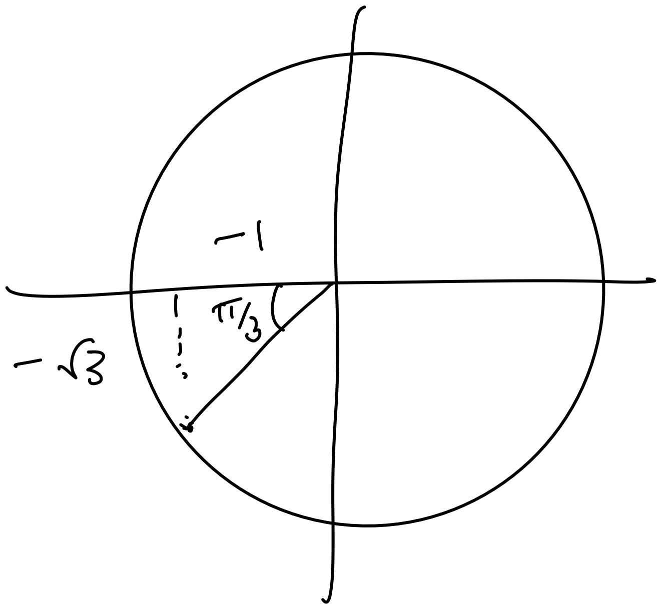
A. $-\sqrt{3}$

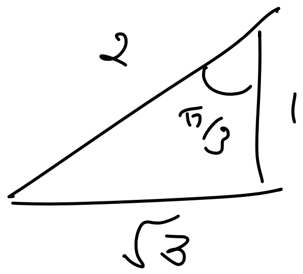
B. $-\frac{\sqrt{3}}{2}$

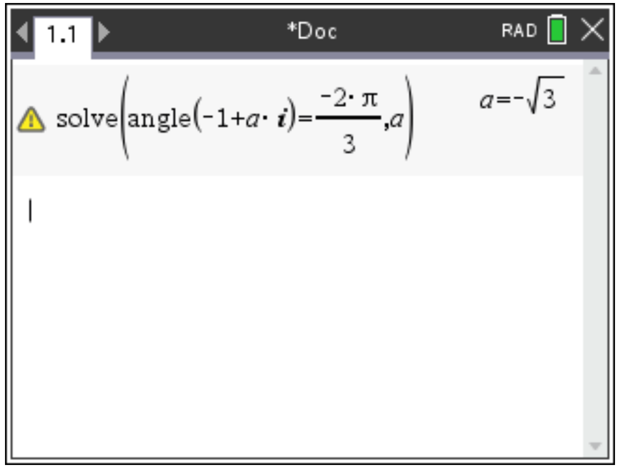
C. $-\frac{1}{\sqrt{3}}$

~~**D.**~~ $\frac{1}{\sqrt{3}}$

~~**E.**~~ $\sqrt{3}$







Learning Objectives: Revisited

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

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Work to be completed

The following represents the minimum work which should be completed.

The more questions you answer from each exercise, chapter review and Checkpoints the better you chance of gaining an excellent study score in November.

Specialist Mathematics Units 3 and 4 Textbook

Chapter 6
Exercise 6C: The modulus-argument form of a complex number
Questions: 1, 2, 3, 4, 5ace, 6bdf, 7, 8ac