# Modulus, conjugate and division

Year 12 Specialist Maths
Units 3 and 4

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## Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what it means to take the module of z
- Understand the properties of the modulus
- Find the conjugate of a complex number
- Know how to divide complex numbers
- Use your CAS to complex the above tasks

# Recap

We have spent the last lesson looking at what a complex number is. We've drawn them on Argand diagrams and know how to add, subtract and multiply them by a scalar.

One of the most important aspects with complex numbers is the idea that multiplying by $i$ is the same as rotating through 90° (in the anti-clockwise direction).

Let's continue to build on this with the new concepts.

$z = a + ib$

$Re(z)$ $Im(z)$

## The modulus … again …

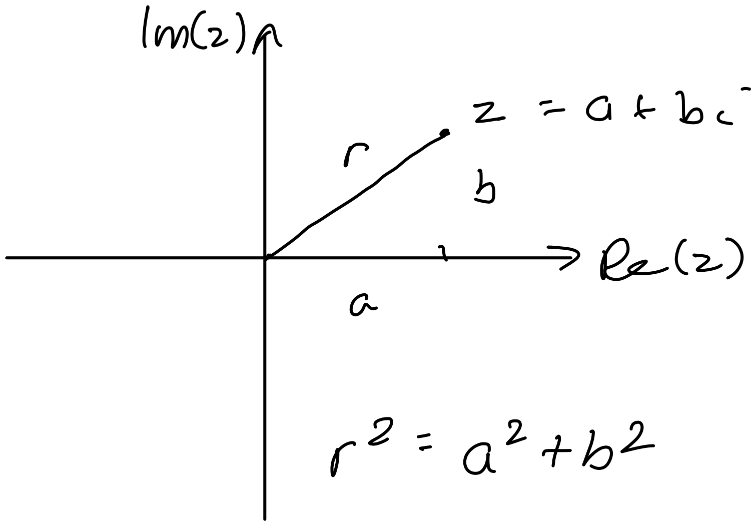
We have already met the concept of a modulus with vectors. In fact, it was even suggested in the previous topic that complex numbers are “the same” as vectors!

Hence, we can find the **distance of the complex number from the origin** by using the following:

$|z| = \sqrt{a^2 + b^2}$ where $z = a + bi$

$|z| = \sqrt{a^2+b^2}$

$z = a + bi$




$r^2 = a^2 + b^2$

$r = \sqrt{a^2+b^2}$

$\underline{\underline{|z| = r}}$

## Properties of the modulus …

It’s never seemingly enough to have just the modulus and move on is it!

Here are some of the most common **and hopefully obvious** properties of the modulus of a complex number.

Remember, this simply represents the distance of the complex number from the origin.

$$|z_1 z_2| = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

This is called the triangle inequality

*Examples have been extracted, with permission, from the Cambridge Mathematical Methods Units 1 and 2 Textbook*

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## The conjugate of a complex number

The conjugate of a complex number is when you make the coefficient of $i$ a negative number. **This is the same as reflecting the complex number in the x-axis**.

i.e. $z = a + bi$ has a conjugate of $\bar{z} = a - bi$

Guess what … these have properties too! Such fun.

$$\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}$$

$$\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$$

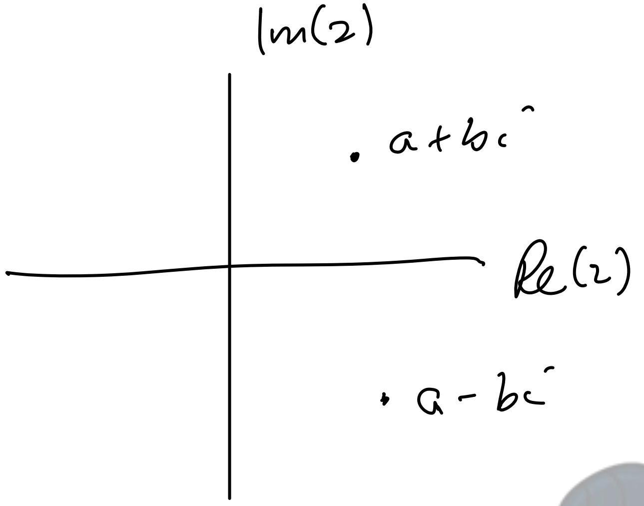
$$\overline{kz} = k\bar{z}$$

$$z\bar{z} = |z|^2$$

$$z + \bar{z} = 2Re(z)$$

$z = a + bi$

$\bar{z} = a - bi$




$z \cdot \bar{z} = |z|^2$

$|z|^2$

$|z| = \sqrt{a^2+b^2}$

$|z|^2 = a^2+b^2$

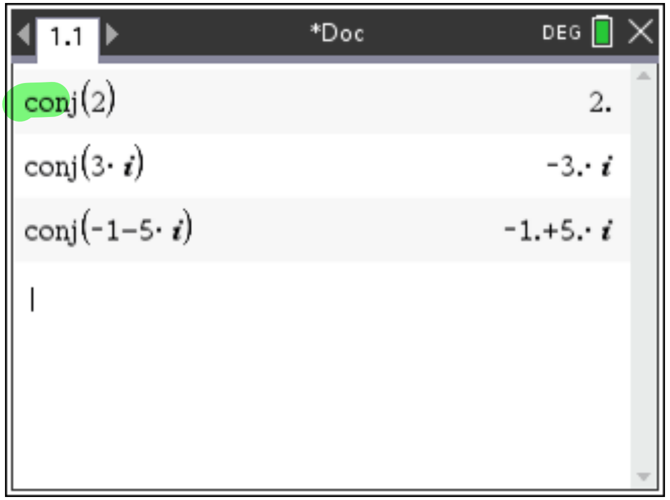
$= [Re(z)]^2 + [Im(z)]^2$

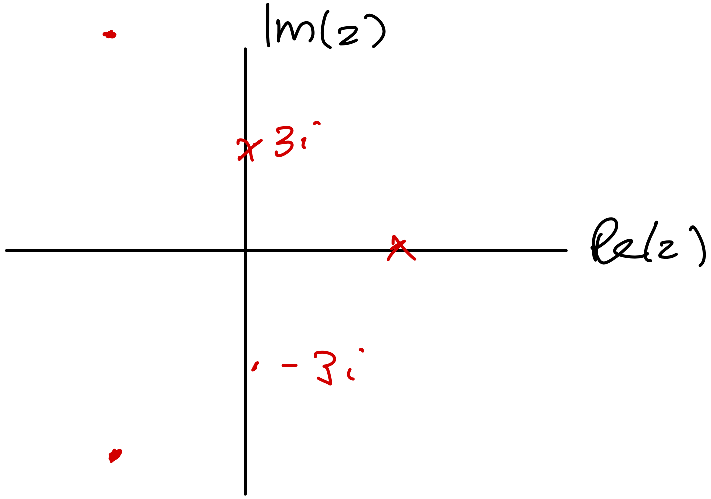
# Examples of finding the conjugate

Find the complex conjugate of each of the following:

- 2 $= 2$
- $3i$ $= -3i$
- $-1-5i$

$-1+5i$




# Division of complex numbers

Dividing complex numbers is much like rationalising the denominator when we have surds. We need to use the complex conjugate to help us "get rid of" the $i$ term(s) we're going to end up with!

Think of this as mixing DOPS with Rationalising the Denominator.

The definition is:

$$\frac{z_1}{z_2} = z_1 z_2^{-1}$$

$$\frac{z_1}{z_2} = z_1 . (z_2)^{-1}$$

$$= \frac{z_1 . \bar{z}}{|z|^2}$$

But what is $z^{-1}$

Let's look at $\frac{1}{z} = \frac{1}{a+bi}$

$$\frac{1}{z} = z^{-1} = \frac{1}{(a+bi)} \frac{a-bi}{(a-bi)}$$

$$= \frac{a-bi}{a^2 - \cancel{abi} + \cancel{abi} - b^2 i^2}$$

$$= \frac{a-bi}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$$

## Division of complex numbers

Hence we can say the following:

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

Barry has decided that this is going to be called the multiplicative inverse of a complex number

## Division of complex numbers

So we can think of division in the following way:

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1\,\overline{z_2}}{|z_2|^2}$$

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z_2}}{|z_2|^2}$$

$$a^2 + b^2 = \mathrm{Re}^2(z) + \mathrm{Im}^2(z)$$

## Example: Division of complex numbers

Write each of the following in the form $a+bi$, where $a,b\in\mathbb{R}$:

- $\dfrac{1}{3-2i}$ $\begin{matrix} z_1 \\ z_2 \end{matrix}$
- $\dfrac{4+i}{3-2i}$

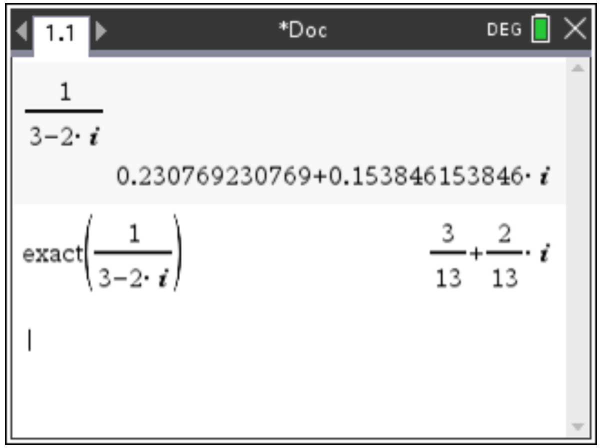
$$\frac{z_1\overline{z_2}}{|z_2|^2}$$

$$\cdot\ \frac{1}{(3-2i)}\cdot\frac{(3+2i)}{(3+2i)}$$

$$=\frac{3+2i}{9+\cancel{6i}-\cancel{6i}-4i^2}$$

$$=\frac{3+2i}{9+4}$$

$$=\frac{3+2i}{13}=\frac{3}{13}+\frac{2i}{13}$$




$$=\frac{1(3+2i)}{13}$$

$$=\frac{3+2i}{13}$$

## Example: Division of complex numbers

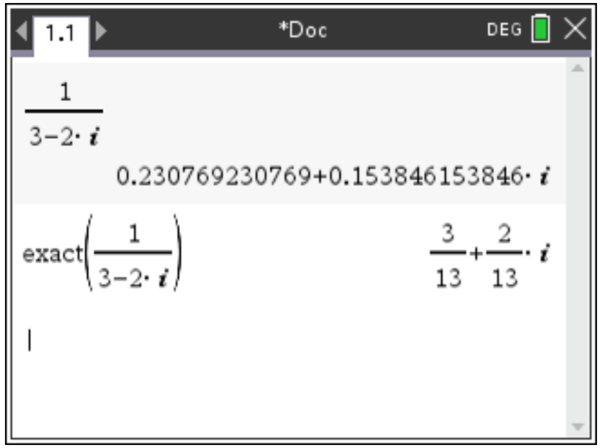
Write each of the following in the form $a+bi$, where $a,b\in\mathbb{R}$:

- $\dfrac{1}{3-2i}$
- $\dfrac{4+i}{3-2i}$

$$= \frac{(4+i)(3+2i)}{9+4}$$

$$= \frac{12+8i+3i+2i^2}{13}$$

$$= \frac{10+11i}{13} = \frac{10}{13}+\frac{11}{13}i$$




$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

# Example: Division of complex numbers

Simplify: $\frac{(1+2i)^2}{i(1+3i)}$

$$= \frac{1+4i+\cancel{4i^2}\ -4}{i+\cancel{3i^2}\ -3}$$

$$\frac{z_1}{z_2} = \ldots$$

$$= \frac{-3+4i}{-3+i}$$

$$= \frac{(-3+4i)(-3-i)}{10}$$

$$= \frac{9+3i-12i-\cancel{4i^2}\ +4}{10} = \frac{13-9i}{10}$$

$$= \frac{13}{10} - \frac{9i}{10}$$

## VCAA Question

**Specialist Maths**
**VCAA 2012 Paper 2**

**Question 8**

If $z = a + bi$, where both $a$ and $b$ are non-zero real numbers and $z \in C$, which of the following does **not** represent a real number?

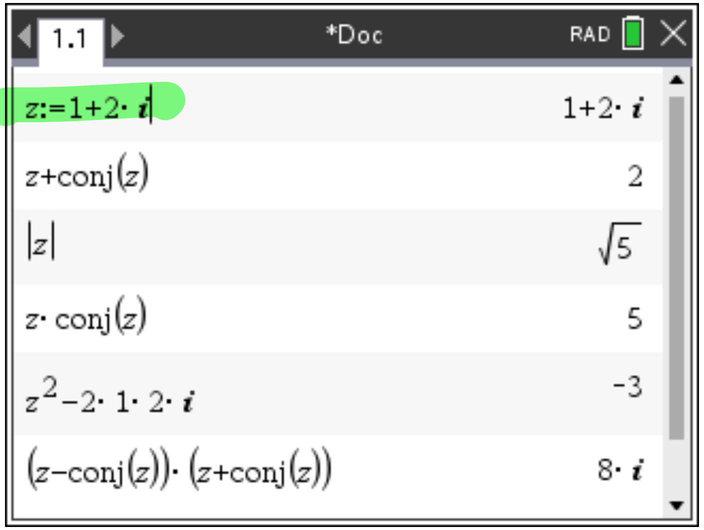
**A.** $z + \overline{z}$

**B.** $|z|$

**C.** $z\overline{z}$

**D.** $z^2 - 2abi$

**E.** $(z - \overline{z})(z + \overline{z})$



## VCAA Question

**Specialist Maths**
**VCAA 2020 Paper 2**

**Question 5**

Given the complex number $z = a + bi$, where $a \in R\backslash\{0\}$ and $b \in R$, $\dfrac{4z\bar{z}}{(z+\bar{z})^2}$ is equivalent to

**A.** $1+\left(\dfrac{\text{Im}(z)}{\text{Re}(z)}\right)^2$

**B.** $4[\text{Re}(z)\times\text{Im}(z)]$

**C.** $4\left([\text{Re}(z)]^2+[\text{Im}(z)]^2\right)$

**D.** $4\left[1+(\text{Re}(z)+\text{Im}(z))^2\right]$

**E.** $\dfrac{2\times\text{Im}(z)}{[\text{Re}(z)]^2}$

$$\overline{(z_1+z_2)} = \overline{z_1}+\overline{z_2}$$

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

$$\overline{kz} = k\bar{z}$$

$$z\bar{z} = |z|^2$$

$$z+\bar{z} = 2Re(z)$$

$$\frac{4z\bar{z}}{(z+\bar{z})^2} = \frac{\cancel{4}\cdot|z|^2}{\cancel{4}\,Re^2(z)}$$

$a^2+b^2$

$Re^2(z) + Im^2(z)$

$$= \frac{Re^2(z)+Im^2(z)}{Re^2(z)}$$

1

$$= \cancel{\frac{Re^2(z)}{Re^2(z)}} + \frac{Im^2(z)}{Re^2(z)}$$

## Learning Objectives: Revisited

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what it means to take the module of z
- Understand the properties of the modulus
- Find the conjugate of a complex number
- Know how to divide complex numbers
- Use your CAS to complex the above tasks

*Examples have been extracted, with permission, from the Cambridge Mathematical Methods Units 1 and 2 Textbook*

# Work to be completed

The following represents the minimum work which should be completed.

The more questions you answer from each exercise, chapter review and Checkpoints the better you chance of gaining an excellent study score in November.

**Specialist Mathematics Units 3 and 4 Textbook**

Chapter 6
Exercise 6B: Modulus, conjugate and division
Questions:  1def, 2cef, 3, 4def, 5abde