Starting to build complex numbers

Year 12 Specialist Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following learning objectives to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand why we need complex numbers
- Understand that $i^2 = -1$
- Understand how to denote a complex number and the set of complex numbers
- Know what Real and Imaginary numbers are

Recap

Another new chapter filled with interesting and exciting things to learn. This chapter is looking at Complex Numbers. We're going to look at what they are, how to use them and their applications in the real world.

$\sqrt{-3}$

As this is the start of a module, there isn't much I can recap. So … let's get on with it.

Note: This is used in a lot of the later topics. So, please treat this as foundational knowledge which you are going to use over and over again.

Negative square roots are actually a thing!

We have been told time and time again that we cannot take the square root of a negative number. Throughout Years 7 to 10 we have happily accepted the fact and just ignored any negative square roots.

Sadly, if there was no such thing as the square root of a negative number, we wouldn't be sitting in the buildings which have been engineered around us.

There are many applications for the square root of a negative number, and we'll get to these in a moment.

For now, it's important to know that if we define $i^2 = -1$ we can use that $i = \sqrt{-1}$

$i^2 = -1$

$i = \sqrt{-1}$

Sets of complex numbers

So … this now opens a whole new number set which we call the complex number set.

A complex number is one which is made up of a real and imaginary component!
Imaginary numbers, unlike imaginary friends, are the component of a number which has the letter i attached.

A complex number is defined as thus:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Imaginary component
i.e. the value of b

Real component
i.e. the value of a

We use the letter z to stand for a complex number.

i.e. $z = a + bi$

The real part of z is denoted as $Re(z)$
The imaginary part of z is denoted as $Im(z)$

$\mathbb{R}$

$z = 3 + 2i$

Real   imaginary

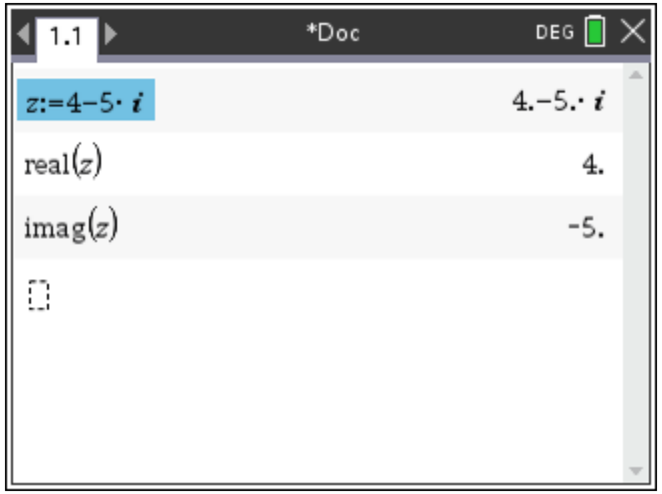
$Re(z)$ $Im(z)$

Example

Let $z = 4 - 5i$. Find:

- $\text{Re}(z)$ 4
- $\text{Im}(z)$ -5
- $\text{Re}(z)$-$\text{Im}(z)$ 9

Notice the :=. This is the same as the define function



$z = 4 - 5i$

$z = 4 - 5i$

$z := 4 - 5i$

$=$

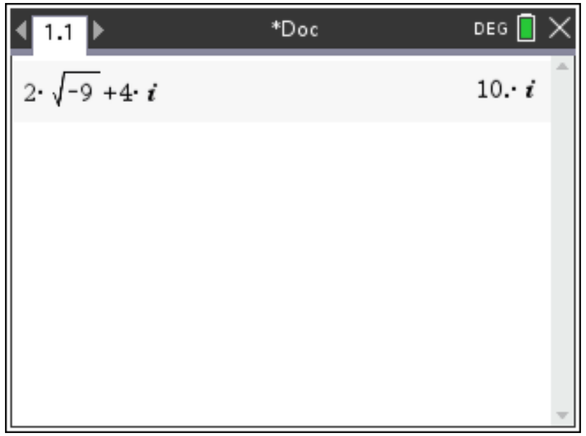
Example

Represent $\sqrt{-5}$ as an imaginary number.

Simplify $2\sqrt{-9} + 4i$.

$$\sqrt{5 \times -1} = \sqrt{5} \times \sqrt{-1}$$
$$= \sqrt{5}\, i$$
$$= i\sqrt{5}$$

$$2\sqrt{-9} + 4i$$
$$2 \times 3i + 4i$$
$$6i + 4i = 10i$$



$$\sqrt{-9} = \sqrt{9 \times -1}$$
$$= \sqrt{9} \times \sqrt{-1}$$
$$= i3$$

Equality of complex numbers

Two complex numbers are said to be equal if both their real and imaginary parts are equal.

Example

Solve the equation $(2a - 3) + 2bi = 5 + 6i$ for $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$$(2a - 3) + 2bi = 5 + 6i$$

$$\therefore \quad 2a - 3 = 5 \qquad 2b\cancel{i} = 6\cancel{i}$$

$$2a = 8 \qquad b = 3$$

$$a = 4$$

Addition and subtraction of complex numbers

When you add and subtract complex numbers you are looking to add the real parts and the imaginary parts.

$z_1 = 2 + 3i$

$z_2 = 3 + 2i$

Examples

Let $z_1 = 2 - 3i$ and $z_2 = 1 + 4i$. Simplify:

- $z_1 + z_2$
- $z_1 - z_2$
- $3z_1 - 2z_2$

$$z_1 + z_2 = 2 - 3i + 1 + 4i$$

$$= \underline{\underline{3 + i}}$$

$$z_1 - z_2 = 2 - 3i - (1 + 4i)$$

$$= 2 - 3i - 1 - 4i$$

$$= \underline{\underline{1 - 7i}}$$

$$3z_1 - 2z_2 = 6 - 9i - (2 + 8i)$$

$$= 6 - 9i - 2 - 8i$$

$$= 4 \underline{\underline{- 17i}}$$

When you multiply a complex number by a scalar you multiply both the real and imaginary parts by the scalar. Remember that you can use your CAS.

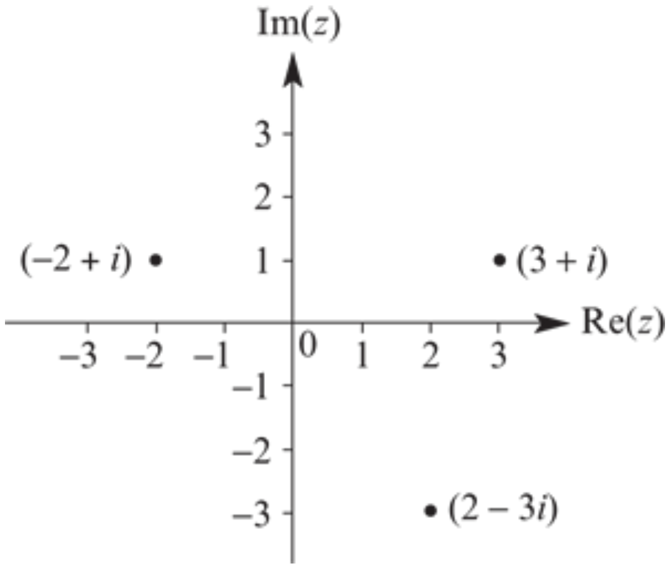
Argand diagrams

Love, love, love and love some more!

These are great as we already have a context when drawing cartesian graphs.

The real part is plotted on the x-axis and the imaginary part is plotted on the y-axis.

Note: A complex number written as $a + bi$ is said to be in **Cartesian form**. These map directly onto a standard cartesian plane. This is really important to know for later sections of the course.

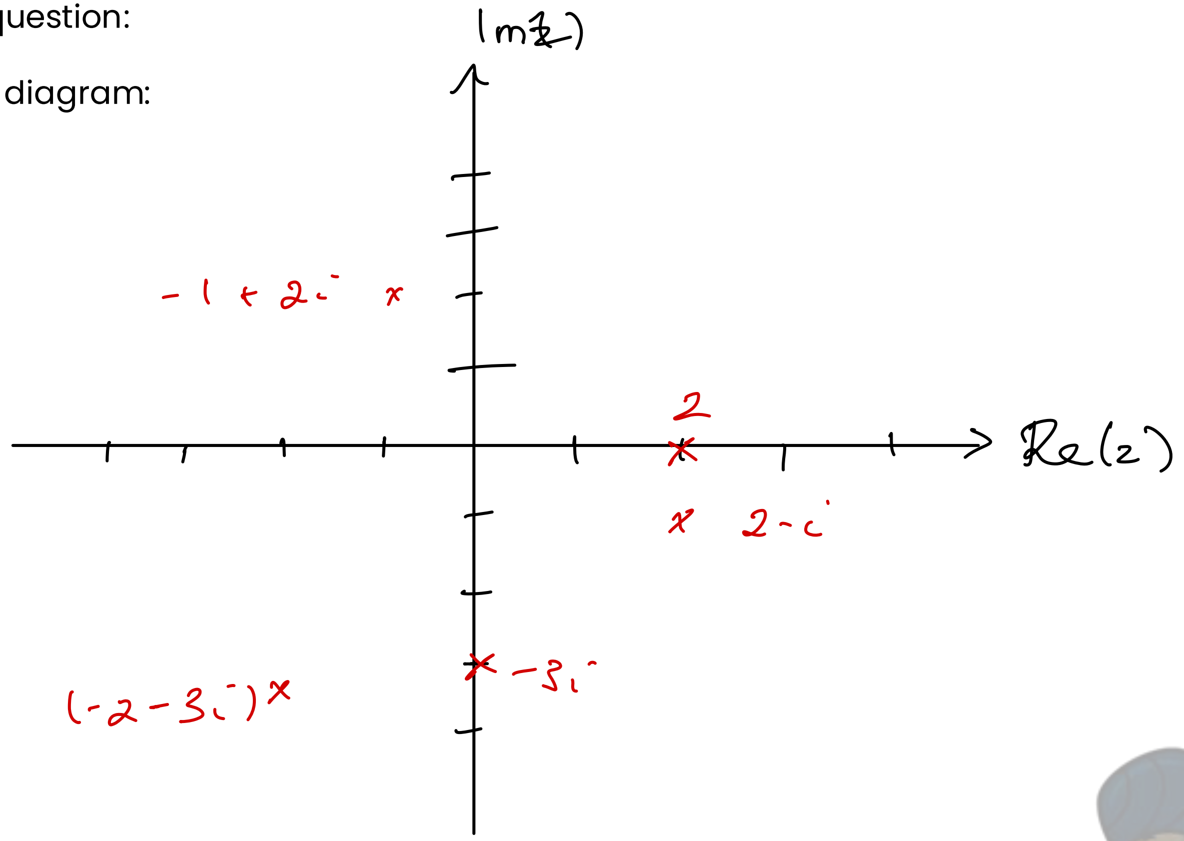


Examples

Let's have a look at drawing an argand diagram for the following question:

Represent the following complex numbers as points on an Argand diagram:

- 2
- $-3i$
- $2-i$
- $-(2+3i)$
- $-1+2i$



Vectors and complex numbers

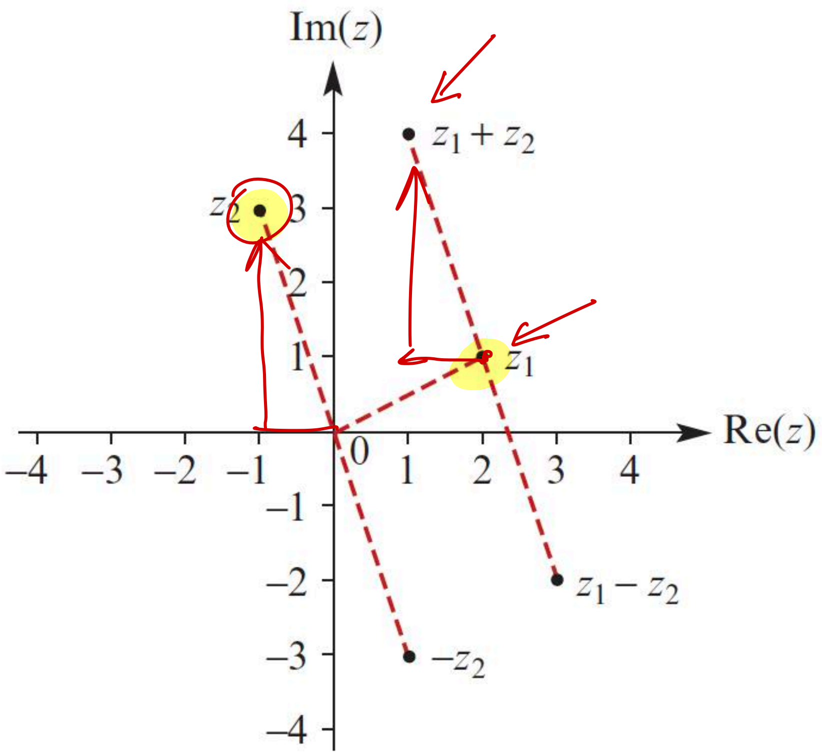
We can think of adding and subtracting complex numbers in much the same way as we do with vectors. We can also extend it to multiplication by a scalar too!

Example:
Let $z_1 = 2 + i$ and $z_2 = -1 + 3i$.

Represent the complex numbers $z_1, z_2, z_1 + z_2$ and $z_1 - z_2$ on an Argand diagram and show the geometric interpretation of the sum and difference.

$Z_1 + Z_2 = 1 + 4i$

$Z_1 - Z_2 =$



Multiplication of complex numbers

$i^2 = -1$

You multiply complex numbers in the same way as you would binomial expansion (FOIL). But, you must remember that $i^2 = -1$. This becomes really important and trips up too many an Mathematician.

We can also use this concept to turn NON-DOPS into DOPS.

Example:

Simplify:

- $(2+3i)(1-5i)$
- $3i(5-2i)$
- i^3

- $(2+3i)(1-5i) = 2 - 10i + 3i - 15i^2$
$= 2 - 10i + 3i - 15(-1)$
$= 2 - 10i + 3i + 15$
$= \underline{\underline{17 - 7i}}$

- $3i(5-2i) = 15i - 6i^2$
$= 15i + 6 = \underline{\underline{6 + 15i}}$

- $i^3 = i^2 . i = -1 . i = \underline{\underline{-i}}$

One of the most important questions

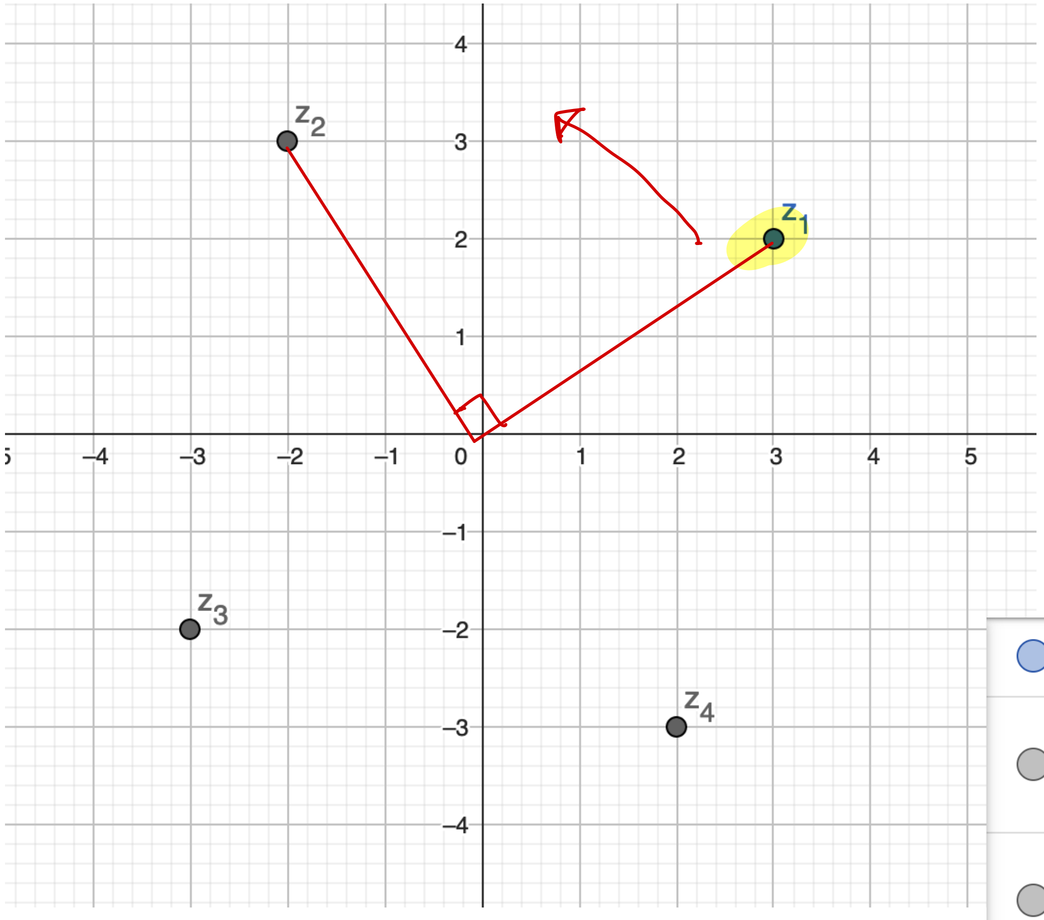
$$i(3+2i)$$
$$= 3i + 2i^2$$
$$= -2 + 3i$$

OK! So … you're probably asking … what does it mean to multiply by i?

Well, let's look at things from an argand diagram point of view.

Start with the complex number $3 + 2i$ and keep multiplying it by i.

What we notice is that multiplying by i means we're rotating **anti-clockwise** by 90°



$z_1 = 3 + 2i$

$z_2 = i\,z_1$
$= -2 + 3i$

$z_3 = i\,z_2$
$= -3 - 2i$

$z_4 = i\,z_3$
$= 2 - 3i$

Powers of *i*

This is a really important concept to understand as it seems to come up on exams over and over again.

- $i^0 = 1$
- $i^1 = i$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = (-1)^2 = 1$
- $i^5 = i$
- $i^6 = -1$
- $i^7 = -i$

In general, for $n = 0, 1, 2, 3, \ldots$

- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

VCAA Question

44%

Specialist Maths
VCAA 2019 Paper 2

Question 4

The expression $i^{1!} + i^{2!} + i^{3!} + \ldots + i^{100!}$ is equal to

A. 0

B. 96

C. $95 + i$

D. $94 + 2i$

E. $98 + 2i$

$i^1 + i^2 + i^6 + i^{24} \ldots$

$i - 1 - 1 + 1 + 1 + 1 + 1 \ldots\ldots$

$i - 2 + 97$

$= i + 95$

$1! = 1$

$2! = 2 \times 1$

$3! = 3 \times 2 \times 1$

$4! = 4 \times 3 \times 2 \times 1$

$i^2 . i^2 . i^2 = -1 . -1 . -1$

$=$

$5! = 5 \times 4 \times 3 \times 2 \times 1$

VCAA Question

Specialist Maths
VCAA 2020 Paper 2

Question 8

Given that $(x + iy)^{14} = a + ib$, where $x, y, a, b \in R$, $(y - ix)^{14}$ for all values of x and y is equal to

A. $-a - ib$

B. $b - ia$

C. $-b + ia$

D. $-a + ib$

E. $b + ia$

$$(ab)^n = a^n b^n$$

$$(y - ix)^{14} = [-i(x + iy)]^{14}$$

$$= (-i)^{14}(x + iy)^{14}$$

$$= -1\,(a + ib)$$

$$= \underline{\underline{-a - ib}}$$

Learning Objectives: Revisited

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$\mathbb{C}$ $z = a + ib$

Work to be completed

The following represents the minimum work which should be completed.

The more questions you answer from each exercise, chapter review and Checkpoints the better you chance of gaining an excellent study score in November.

Specialist Mathematics Units 3 and 4 Textbook

Chapter 6: Questions 7 to 11 inclusive

Note: If you feel you would like practice as the more basic concepts, please start at an earlier question.