



Vector projections

Year 12 Specialist Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what a vector projection or vector resolute is
- Understand the term rectangular components.



Recap

This is the next lesson in this section of the course looking at Vectors. Whilst it's coming to a close, the work will be used throughout the rest of the course and hence it is vital to ensure that the understanding is complete.

We have already looked at how we can find the angles between two vectors using the scalar (or dot) product.

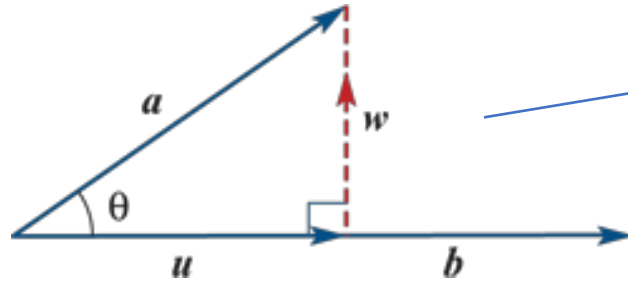
This can be used to find the shortest distance between two vectors by using something called vector projections.



Decomposing vectors

It is often useful to decompose a vector into the sum of two vectors; one parallel to a given vector and the other perpendicular to the given vector.

For example we can see that $a = u + w$



Vector w represents the shortest distance between the end of vector a and vector b .

Apparently, you can use this to find the closest point between two vectors

We can also see $u = kb$

$$\underline{a} = \underline{u} + \underline{w}$$

$$\underline{w} \cdot \underline{b} = 0$$

$$\underline{u} = k \underline{b}$$

$$\underline{w} = \underline{a} - \underline{u}$$

$$\underline{w} = \underline{a} - k \underline{b}$$

$$(\underline{a} - k \underline{b}) \cdot \underline{b} = 0$$

$$\underline{a} \cdot \underline{b} - k \underline{b} \cdot \underline{b} = 0$$

$$k \underline{b} \cdot \underline{b} = \underline{a} \cdot \underline{b}$$

$$k = \frac{\underline{a} \cdot \underline{b}}{\underline{b} \cdot \underline{b}}$$

$$\underline{u} = \left(\frac{\underline{a} \cdot \underline{b}}{\underline{b} \cdot \underline{b}} \right) \underline{b}$$



Decomposing vectors: Another way

It might make more sense to look at the ideas again by using Trigonometry and the work we have already covered. We know the length of vector u can be found using Trig:

$$|u| = |a| \cos \theta$$

We know that we can find the angle using

$$\cos \theta = \frac{a \cdot b}{|a||b|}$$

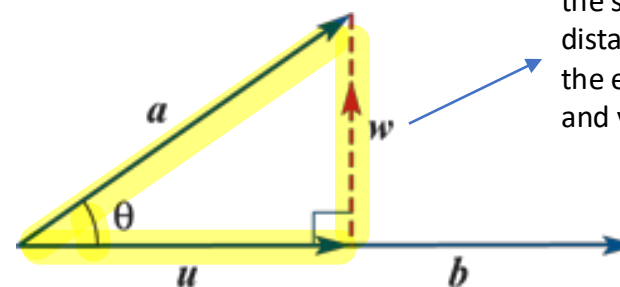
Hence, length of vector u is $|u| = \frac{a \cdot b}{|a||b|} |a|$

Which simplifies to $|u| = \frac{a \cdot b}{|b|}$ this can be expressed as $|u| = a \cdot \hat{b}$

$$\cos \theta = \frac{|u|}{|a|}$$

$$|u| = |a| \cos \theta$$

$$|a| \left(\frac{a \cdot b}{|a||b|} \right)$$



Vector w represents the shortest distance between the end of vector a and vector b .

$$|u| = \left(\frac{a \cdot b}{|a||b|} \right) |a| = \frac{\cancel{|a|} a \cdot b}{\cancel{|a|} |b|} = \frac{a \cdot b}{|b|} = a \cdot \hat{b}$$

$|u|$ is the length of the vector u . It can also be called the **scalar resolute**. The scalar resolute can also be referred to as the magnitude of the vector projection.



Vector resolute

The vector u is called the **vector projection** or **vector resolute** of a in the direction of b .

The vector resolute of a in the direction of b can be expressed in any one of the following forms:

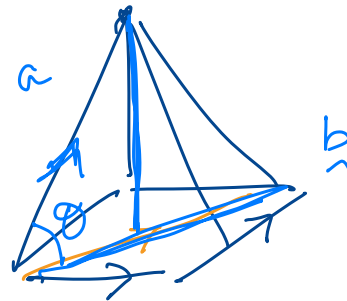
$$u = \frac{a \cdot b}{b \cdot b} b = \frac{a \cdot b}{|b|^2} b = \left(a \cdot \frac{b}{|b|} \right) \left(\frac{b}{|b|} \right) = (a \cdot \hat{b}) \hat{b}$$

This is the form we've just looked at

The above can be found using $k(a \cdot b) = (ka) \cdot b = a \cdot (kb)$

And $b \cdot b = |b|^2$

The quantity $(a \cdot \hat{b}) = \frac{a \cdot b}{|b|}$ is called the **signed length** of the vector resolute u and is called the **scalar resolute** of a in the direction of b



$$\hat{u} = \left(\frac{a \cdot b}{b \cdot b} \right) b$$

$$\hat{u} = \left(\frac{a \cdot b}{|b|^2} \right) b$$

$$= \frac{(a \cdot b) b}{|b| |b|}$$

$$= \left(\frac{a \cdot b}{|b|} \right) \hat{b}$$

$$= (a \cdot \hat{b}) \hat{b}$$



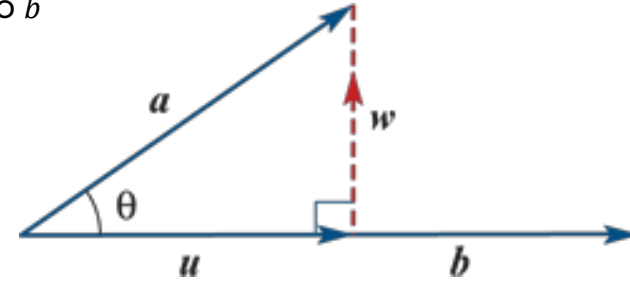
Rectangular components

Knowing that we can express a in terms of u and w means we can express a as the sum of the two components, the first parallel to b and the second perpendicular to b gives:

$$a = \frac{(a \cdot b)}{(b \cdot b)} b + \left(a - \frac{a \cdot b}{b \cdot b} b \right)$$

Horizontal component
which is u

Vertical component which is w



$$w = a - u$$

$$u = \left(\frac{a \cdot b}{b \cdot b} \right) b$$

$$a = u + w$$

$$a = u + (a - u)$$

$$a = \left(\frac{a \cdot b}{b \cdot b} \right) b + \left(a - \frac{a \cdot b}{b \cdot b} b \right)$$



Example

Let $a = i + 3j - k$ and $b = i - j + 2k$. Find the vector resolute of:

- a in the direction of b
- b in the direction of a .

$$u = \frac{a \cdot b}{b \cdot b} b = \frac{a \cdot b}{|b|^2} b = \left(a \cdot \frac{b}{|b|} \right) \left(\frac{b}{|b|} \right) = (a \cdot \hat{b}) \hat{b}$$



$$u = \left(\frac{a \cdot b}{b \cdot b} \right) b$$

$$= \frac{-2}{3} i + \frac{2}{3} j - \frac{4}{3} k$$

$$u = \left(\frac{a \cdot b}{a \cdot a} \right) a = \frac{-4}{11} i + \frac{12}{11} j + \frac{4}{11} k$$

1.1 *Doc DEG

$a := [1 \ 3 \ -1]$	$[1 \ 3 \ -1]$
$b := [1 \ -1 \ 2]$	$[1 \ -1 \ 2]$
$\frac{\text{dotP}(a,b)}{\text{dotP}(b,b)} \cdot b$	$\begin{bmatrix} -2 & 2 & -4 \\ 3 & 3 & 3 \end{bmatrix}$
$\frac{\text{dotP}(b,a)}{\text{dotP}(a,a)} \cdot a$	$\begin{bmatrix} -4 & -12 & 4 \\ 11 & 11 & 11 \end{bmatrix}$
$[\]$	

1.1 *Doc RAD

$a := [1 \ 3 \ -1]$	$[1. \ 3. \ -1.]$
$b := [1 \ -1 \ 2]$	$[1. \ -1. \ 2.]$
$\text{dotP}(a, \text{unitV}(b)) \cdot \text{unitV}(b)$	$\begin{bmatrix} -2 & 2 & -4 \\ 3 & 3 & 3 \end{bmatrix}$
$\text{dotP}(b, \text{unitV}(a)) \cdot \text{unitV}(a)$	$\begin{bmatrix} -4 & -12 & 4 \\ 11 & 11 & 11 \end{bmatrix}$

Example

Find the scalar resolute of $a = 2i + 2j - k$ in the direction of $b = -i + 3k$:

$$u = \frac{a \cdot b}{b \cdot b} b = \frac{a \cdot b}{|b|^2} b = \left(a \cdot \frac{b}{|b|} \right) \left(\frac{b}{|b|} \right) = (a \cdot \hat{b}) \hat{b}$$

$$|u| = \frac{a \cdot b}{|b|} = \frac{-\sqrt{10}}{2}$$

TI-84 Plus calculator screen (DEG mode) showing the calculation of the scalar resolute of vector a in the direction of vector b . The screen displays the following steps:

- Define vector a : $a := [2 \ 2 \ -1]$
- Define vector b : $b := [-1 \ 0 \ 3]$
- Calculate the dot product: $\text{dotP}(a, b)$
- Calculate the magnitude of b : $\text{norm}(b)$
- The result of the division is $\frac{-\sqrt{10}}{2}$.

TI-84 Plus calculator screen (RAD mode) showing the calculation of the scalar resolute of vector a in the direction of vector b . The screen displays the following steps:

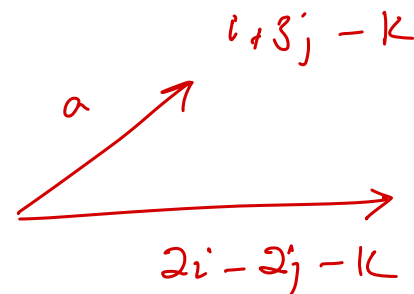
- Define vector a : $a := [2 \ 2 \ -1]$
- Define vector b : $b := [-1 \ 0 \ 3]$
- Calculate the dot product: $\text{dotP}(a, \text{unitV}(b))$
- The result of the calculation is $\frac{-\sqrt{10}}{2}$.

Example

Resolve $i + 3j - k$ into rectangular components, one of which is parallel to $2i - 2j - k$

Remember:

$$u = \frac{a \cdot b}{b \cdot b} b = \frac{a \cdot b}{|b|^2} b = \left(a \cdot \frac{b}{|b|} \right) \left(\frac{b}{|b|} \right) = (a \cdot \hat{b}) \hat{b}$$



$$a = \frac{a \cdot b}{b \cdot b} \cdot \underline{b} + \left(a - \frac{a \cdot b}{b \cdot b} \underline{b} \right)$$

$$a = \left(-\frac{2}{3}i + \frac{2}{3}j + \frac{1}{3}k \right) + \left(\frac{5}{3}i + \frac{7}{3}j - \frac{4}{3}k \right)$$

```
1.1 *Doc RAD
a:=[1 3 -1] [1 3 -1]
b:=[2 -2 -1] [2 -2 -1]
dotP(a,unitV(b)).unitV(b) [-2/3 2/3 1/3]
a-[-2/3 2/3 1/3] [5/3 7/3 -4/3]
```



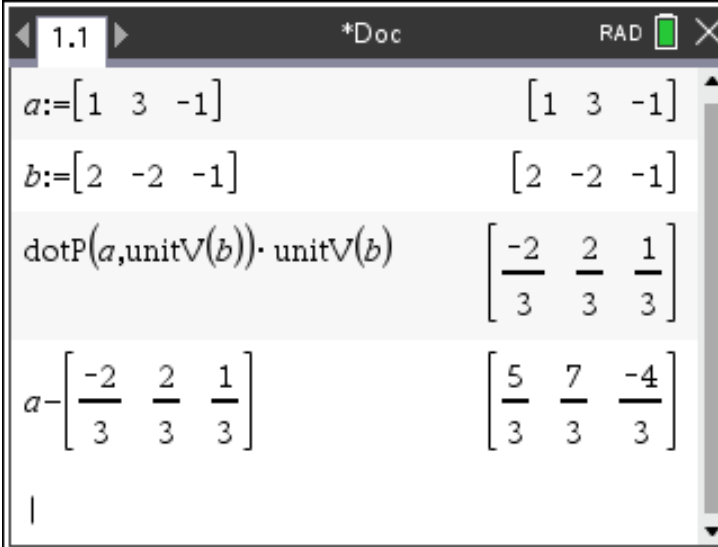
Example

Resolve $i + 3j - k$ into rectangular components, one of which is parallel to $2i - 2j - k$

Remember:

$$u = \frac{a \cdot b}{b \cdot b} b = \frac{a \cdot b}{|b|^2} b = \left(a \cdot \frac{b}{|b|} \right) \left(\frac{b}{|b|} \right) = (a \cdot \hat{b}) \hat{b}$$

Hence, we can write a as $-\frac{1}{3}(2i - 2j - k) + \frac{1}{3}(5i + 7j - 4k)$



The calculator screen shows the following steps and results:

Input	Output
$a := [1 \ 3 \ -1]$	$[1 \ 3 \ -1]$
$b := [2 \ -2 \ -1]$	$[2 \ -2 \ -1]$
$\text{dotP}(a, \text{unitV}(b)) \cdot \text{unitV}(b)$	$\begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$
$a - \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$	$\begin{bmatrix} \frac{5}{3} & \frac{7}{3} & -\frac{4}{3} \end{bmatrix}$

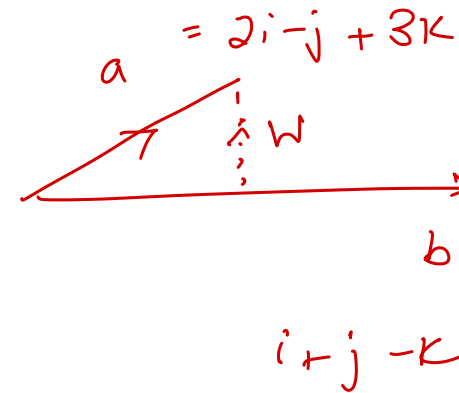


VCAA Questions

Question 11

The vector resolute of $\underline{a} = 2\underline{i} - \underline{j} + 3\underline{k}$ that is **perpendicular** to $\underline{b} = \underline{i} + \underline{j} - \underline{k}$ is

- A. $-\frac{2}{3}(\underline{i} + \underline{j} - \underline{k})$
- B. $-\frac{2}{3}(2\underline{i} - \underline{j} + 3\underline{k})$
- C. $\frac{1}{3}(8\underline{i} - \underline{j} + 7\underline{k})$**
- D. $\underline{i} - 2\underline{j} + 4\underline{k}$
- E. $\underline{i} + \underline{j} + 2\underline{k}$



$$\underline{a} - (\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}}$$

Specialist Maths 2019NHT Paper 2

$$\underline{a} = \frac{(\underline{a} \cdot \underline{b})}{(\underline{b} \cdot \underline{b})} \underline{b} + \left(\underline{a} - \frac{\underline{a} \cdot \underline{b}}{\underline{b} \cdot \underline{b}} \underline{b} \right)$$

$$\underline{a} = \frac{(\underline{a} \cdot \underline{b})}{(\underline{b} \cdot \underline{b})} \underline{b} + \underline{\underline{\left(\underline{a} - (\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}} \right)}}$$



VCAA Questions

Question 11

The vector resolute of $\underline{a} = 2\underline{i} - \underline{j} + 3\underline{k}$ that is **perpendicular** to $\underline{b} = \underline{i} + \underline{j} - \underline{k}$ is

- A. $-\frac{2}{3}(\underline{i} + \underline{j} - \underline{k})$
- B. $-\frac{2}{3}(2\underline{i} - \underline{j} + 3\underline{k})$
- C. $\frac{1}{3}(8\underline{i} - \underline{j} + 7\underline{k})$
- D. $\underline{i} - 2\underline{j} + 4\underline{k}$
- E. $\underline{i} + \underline{j} + 2\underline{k}$

Specialist Maths 2019NHT Paper 2

```
1.1 *Doc RAD
a:=[2 -1 3] [2 -1 3]
b:=[1 1 -1] [1 1 -1]
a-dotP(a,unitV(b)).unitV(b) [8/3 -1/3 7/3]
```



VCAA Questions

Specialist Maths
2018NHT Paper 1

Question 2 (3 marks)

Let $\underline{a} = 3\underline{i} - 2\underline{j} + m\underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + 3\underline{k}$, where $m \in \mathbb{R}$.

Find the value(s) of m such that the magnitude of the vector resolute of $\underline{a}$ parallel to $\underline{b}$ is equal to $\sqrt{14}$.

$$\hat{\underline{b}} = \frac{1}{\sqrt{14}} (2\underline{i} - \underline{j} + 3\underline{k}) = \frac{\sqrt{14}}{14} (2\underline{i} - \underline{j} + 3\underline{k})$$

$$\underline{a} \cdot \hat{\underline{b}} = \frac{\sqrt{14}}{14} (6 + 2 + 3m) = \frac{\sqrt{14}}{14} (8 + 3m)$$

$$\frac{\sqrt{14}}{14} (8 + 3m) \frac{\sqrt{14}}{14} (2\underline{i} - \underline{j} + 3\underline{k}) = \frac{1}{14} (8 + 3m) (2\underline{i} - \underline{j} + 3\underline{k})$$

$$|(\underline{a} \cdot \hat{\underline{b}}) \hat{\underline{b}}| = \frac{(8 + 3m) \sqrt{14}}{14} = \pm \sqrt{14}$$

$$8 + 3m = \pm 14$$

$$3m = \pm 14 - 8$$

$$\underline{u} = \frac{\underline{a} \cdot \underline{b}}{\underline{b} \cdot \underline{b}} \underline{b} = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|^2} \underline{b} = \left(\underline{a} \cdot \frac{\underline{b}}{|\underline{b}|} \right) \left(\frac{\underline{b}}{|\underline{b}|} \right) = (\underline{a} \cdot \hat{\underline{b}}) \hat{\underline{b}}$$

$$\begin{aligned} 3m &= -22 \\ m &= -22/3 \\ 3m &= 6 \end{aligned}$$

$$m = 2$$



Learning Objectives: Revisited

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what a vector projection or vector resolute is
- Understand the term rectangular components.



Work to be completed

The following represents the minimum work which should be completed.

The more questions you answer from each exercise, chapter review and Checkpoints the better your chance of gaining an excellent study score in November.

Specialist Mathematics Units 3 and 4 Textbook

Chapter 4

Exercise 4D: Vector projections

Questions: 1, 3, 4, 7, 8, 9 and 10

