Scalar product of vectors

Year 12 Specialist Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what the scalar product is
- Understand what the geometric description of the scalar product is
- Know the properties of the scalar product
- Find the magnitude of the angle between two vectors

Recap

In this third lesson in the series dealing with vectors we build on the ideas from the past lessons relating to finding the magnitude of vectors.

I have been asked over and over again ….

"What is the scalar product? Like really??"

I quote …

I take the following from the Cambridge Textbook … and they are normally very good at describing what something is!

> *“The scalar product is an operation that takes two vectors and gives a real number.”*

I even looked it up in a dictionary …

From merriam-webster.com

scalar product noun

Save Word

Definition of *scalar product*

: a real number that is the product of the lengths of two vectors and the cosine of the angle between them

— called also *dot product*, *inner product*

Otherwise known as … the dot product!

The scalar product of two vectors is defined as:

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

It can be used to help us prove the compound angle formulae (in Chapter 3) but other than that … I've not yet found a practical application!

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

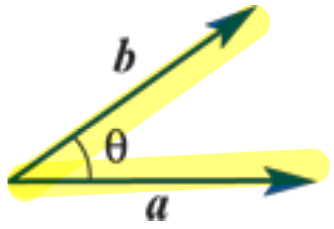
$$a = a_1 i + a_2 j + a_3 k$$

$$b = b_1 i + b_2 j + b_3 k$$

Geometric description of the scalar product

If we have vectors a and b then we have:

$$a \cdot b = |a||b| \cos\theta$$

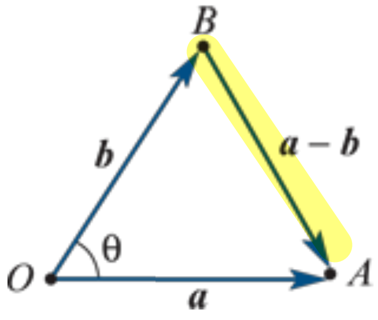


It's important to note that the angle between the vectors is the one when the vectors are "**pointing away from the point of connection**". This again becomes important in Chapter 3 and the Unit Circle.

$$|a-b|^2 = |a|^2 + |b|^2 - 2.|a||b|.\cos\theta$$

$$(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2 = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2|a||b|.\cos\theta$$

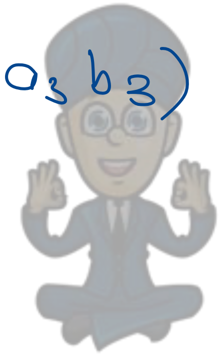
Which can be shown using cosine rule and the following triangle:



$$\cancel{2}|a||b|.\cos\theta = \cancel{2}(a_1b_1 + a_2b_2 + a_3b_3)$$

$$|a||b|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$$

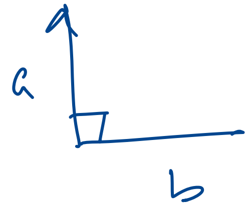
$$|a||b|\cos\theta = a \cdot b$$



Properties of the scalar product

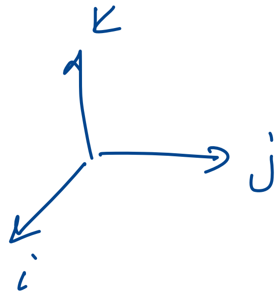
More properties to know and remember!

$a \cdot b = b \cdot a$
$k(a \cdot b) = (ka) \cdot b = a \cdot (kb)$
$a \cdot \mathbf{0} = \mathbf{0}$
$a \cdot (b + c) = a \cdot b + a \cdot c$
$a \cdot a = |a|^2$



If the vectors a and b are perpendicular then $a \cdot b = 0$

For parallel vectors a and b we have $\mathrm{a} \cdot b = \begin{cases} |a||b| & if\, a\ and\ b\ are\ parallel\ and\ in\ the\ same\ direction \\ -|a||b|. & if\ a\ and\ b\ are\ parallel\ and\ in\ opposite\ directions \end{cases}$



For the unit vectors i, j and k we know that $i \cdot i = j \cdot j = k \cdot k = 1$

For the unit vectors i, j and k we know that $i \cdot j = j \cdot k = j \cdot k = 0$

Example

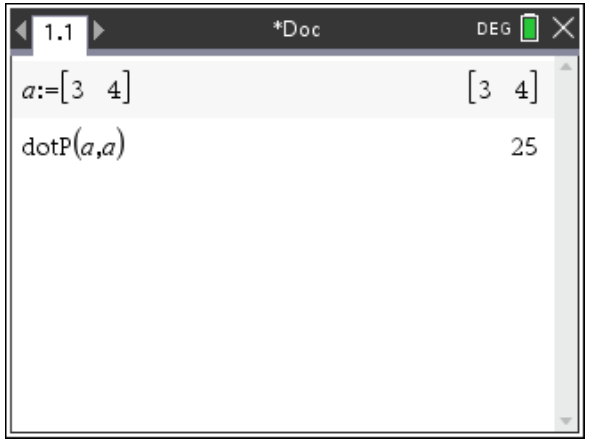
Let $a = i - 2j + 3k$ and $b = -2i + 3j + 4k$. Find:

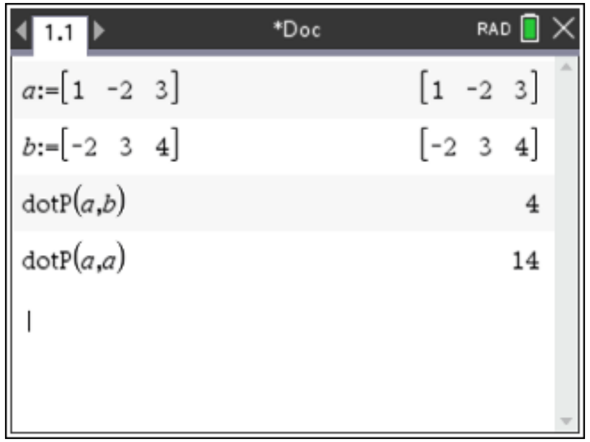
- $a \cdot b$
- $a \cdot a$

Use pencil and paper and the CAS.

Note: the dot product of a vector with itself is the length squared!

$$a \cdot a = (1)(1) + (-2)(-2) + (3)(3)$$
$$= 1 + 4 + 9$$
$$= \underline{\underline{14}}$$

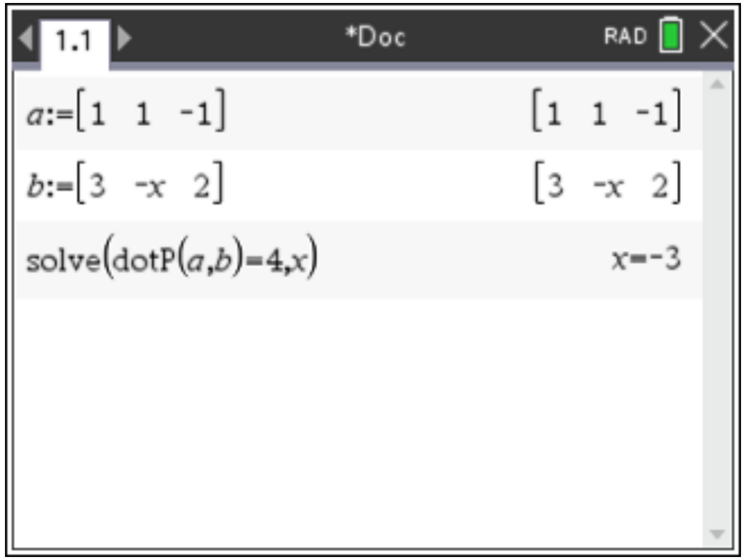




$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$
$$= 1(-2) + (-2)(3) + (3)(4)$$
$$= -2 - 6 + 12$$
$$= \underline{\underline{4}}$$

Example

Solve the equation $(i + j - k) \cdot (3i - xj + 2k) = 4 \; for \; x$



$$1(3) + (1)(-x) + (-1)(2) = 4$$

$$3 - x - 2 = 4$$

$$3 - 2 - 4 = x$$

$$x = \underline{\underline{-3}}$$

Finding the magnitude of the angle between two vectors

norm( )

We can find the angle using the previously given formula:

$$a \cdot b = |a||b| \cos \theta$$

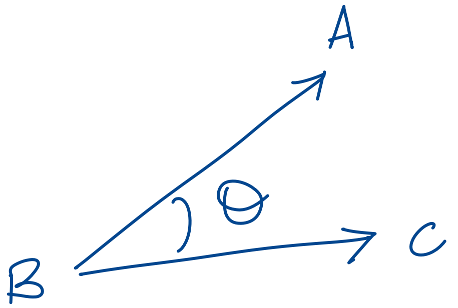
Hence

$$\cos \theta = \frac{a \cdot b}{|a||b|}$$

Note: The angle always connects the tails of the vectors. i.e. the arrows point away.

$\angle ABC$

Hence, $\angle ABC$ can be found using $\overrightarrow{BA}$ and $\overrightarrow{BC}$

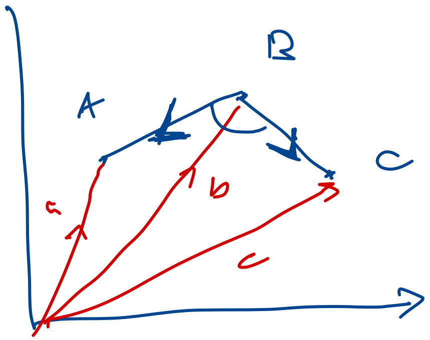


Example

A, B and C are points defined by the position vectors a, b *and* c respectively, where

$a = i + 3j - k$, $b = 2i + j$ and $c = i - 2j - 2k$

Find the magnitude of $\angle ABC$, correct to one decimal place.



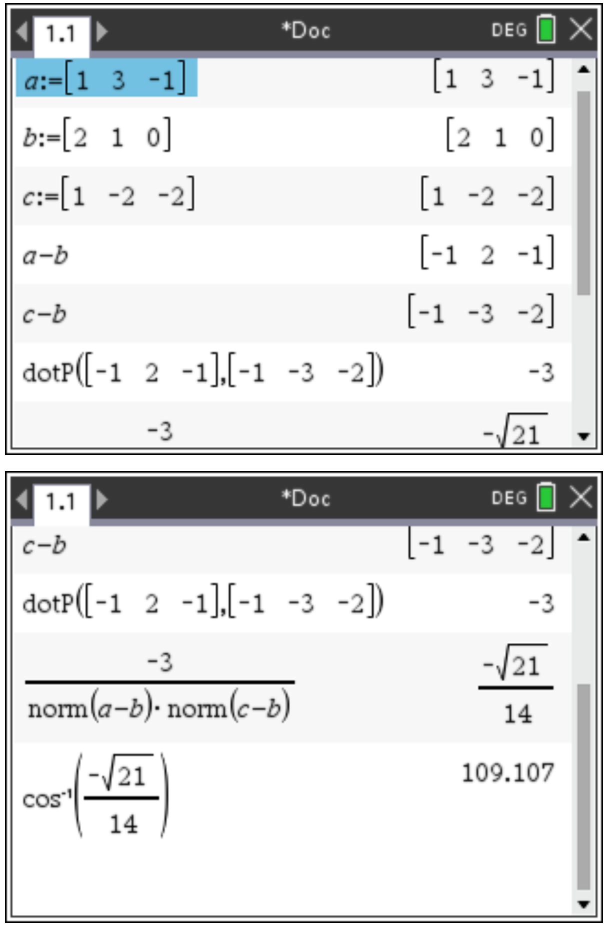
$\overrightarrow{BA} = a - b$

$\overrightarrow{BC} = c - b$

$\cos\theta = \dfrac{BA \cdot BC}{|BA||BC|}$

$\cos\theta = -\dfrac{\sqrt{21}}{14}$

$\therefore \theta = \underline{\underline{109.1^\circ}}$



VCAA Question

Specialist Mathematics
2020 Paper 2

Question 16

Let $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}$, where the acute angle between these vectors is θ.

The value of $\sin(2\theta)$ is

A. $\dfrac{1}{9}$

B. $\dfrac{4\sqrt{5}}{9}$

C. $\dfrac{4\sqrt{5}}{81}$

D. $\dfrac{8\sqrt{5}}{81}$

E. $\dfrac{2\sqrt{46}}{25}$

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{1}{9}$$

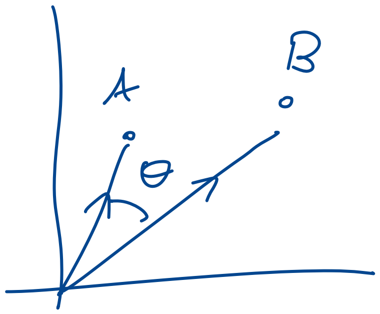
$$\sin\theta = 2 \cdot \frac{\sqrt{80}}{9} \cdot \frac{1}{9}$$

$$= \frac{2\sqrt{80}}{81}$$

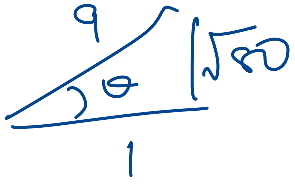
$$= \frac{8\sqrt{5}}{81}$$

$$\sqrt{80} = \sqrt{16 \times 5}$$

$$= 4\sqrt{5}$$



$$\sin 2\theta = 2\sin\theta\cos\theta \checkmark$$



$$9^2 - 1^2 = \sqrt{80}$$

Learning Objectives: Revisited

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Work to be completed

The following represents the minimum work which should be completed.

The more questions you answer from each exercise, chapter review and Checkpoints the better you chance of gaining an excellent study score in November.

Specialist Mathematics Units 3 and 4 Textbook

Chapter 2
Exercise 2C: Scalar product of vectors
Questions: 2, 4, 5, 8, 10, 11, 14, 16