Resolution of vectors into rectangular components

Year 12 Specialist Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what is meant by a unit vector
- Understand the I, j and k notation
- Find the magnitude of a vector in both 2 and 3-dimensions
- Be able to add and subtract vectors in 2- and 3-dimensions
- Find the angle made by a vector with an axis

Recap

In this, the second video dealing with Vectors for the Specialist Mathematics Units 3 and 4 course, we continue looking at Vectors.

In the previous video we took a look at the basics of vectors jumping very quickly into the idea of linear dependence and independence. Remember, when two or more vectors are **linearly dependent** then one vector can be expressed as a sum of the other two vectors (which may, or may not have been scaled).

This lesson is going to build on the work covered in the previous video.

A Unit Vector

When I was at school doing this (147 years ago) … I had no idea what the unit vector was. My teacher made such a meal of it I was forever confused. Thankfully I've undone years of heartache and found it to be a lot easier than I was taught it!

A **unit vector** is a vector in the same direction but with **one unit length**. This small fact means I can use Pythagoras to help me find it!

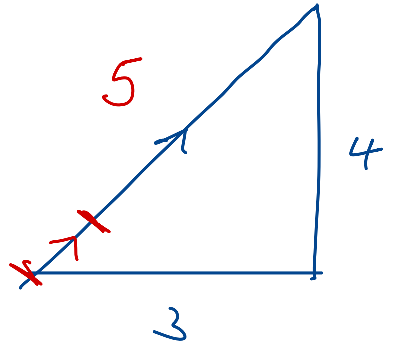
For example, look at the vector below.

$$a = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

We know that the magnitude of this vector is $|a| = \sqrt{3^2 + 4^2} = 5$

Hence, the magnitude of this vector is 5 units. If I wanted a vector in the same direction as a but only **one unit** long, I'd simply multiply the vector by $\frac{1}{5}$

Note: This builds on the idea of midpoints of lines etc.



$$\sqrt{3^2 + 4^2}$$

$$\hat{a} = \frac{1}{|a|} \cdot \underset{\sim}{a}$$

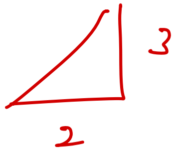
Unit Vector Notation

The unit vector of a is denoted as $\hat{a}$ (a hat)

Knowing the unit vector we can then use it to find a vector of any length in the direction of a

So, we know:

$$\hat{a} = \frac{1}{|a|}\underset{\sim}{a}$$

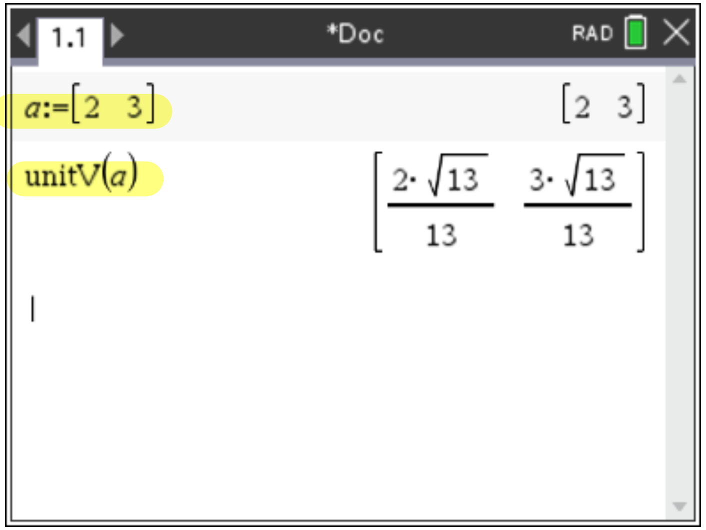


$$|a| = \sqrt{2^2 + 3^2}$$

$$= \sqrt{13}$$

$$\hat{a} = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}$$



We can use the CAS to find a unit vector.

i, j and k notation

It's going to be useful to split vectors up into their respective distances along the traditional axes. You could argue we have already been doing this with column vector notation.

Hence, the convention is:

The unit vector in the positive direction of the x-axis is i.
The unit vector in the positive direction of the y-axis is j.
The unit vector in the positive direction of the z-axis is k.

Note: The unit vectors I, j and k are linearly independent.

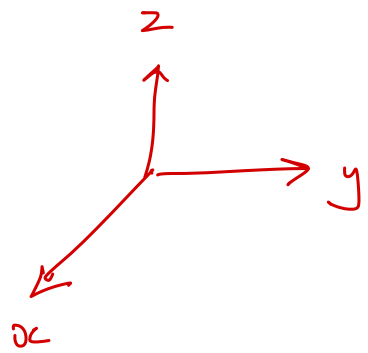
For the point $R(x, y)$ we can express it as $\overrightarrow{OR} = xi + yj$

For the point S(x, y, z) we can express it as $\overrightarrow{OS} = xi + yj + zk$

i

j

k



Adding and subtracting in i, j and k notation

Much like we did before with column vectors, when adding with i, j *and* k notation, we simply add the i values together, the j values together and the k values together.

Easy as!

$\underset{\sim}{a} = i + 2j + 3k$

$\underset{\sim}{b} = i - j + 4k$

$\underset{\sim}{a} + \underset{\sim}{b} = 2i + j + 7k$

Example

Using the vectors i and j, give the vectors:

$\overrightarrow{OA}$ $2i + 3j$ $-2i + 3j$

$\overrightarrow{OB}$ $4i + j$

$\overrightarrow{OC}$ $i - 2j$

$\overrightarrow{OD}$

Using the vectors i and j, give the vectors:

$\overrightarrow{AB}$ $2i - 2j$

$\overrightarrow{BC}$ $-3i - 3j$

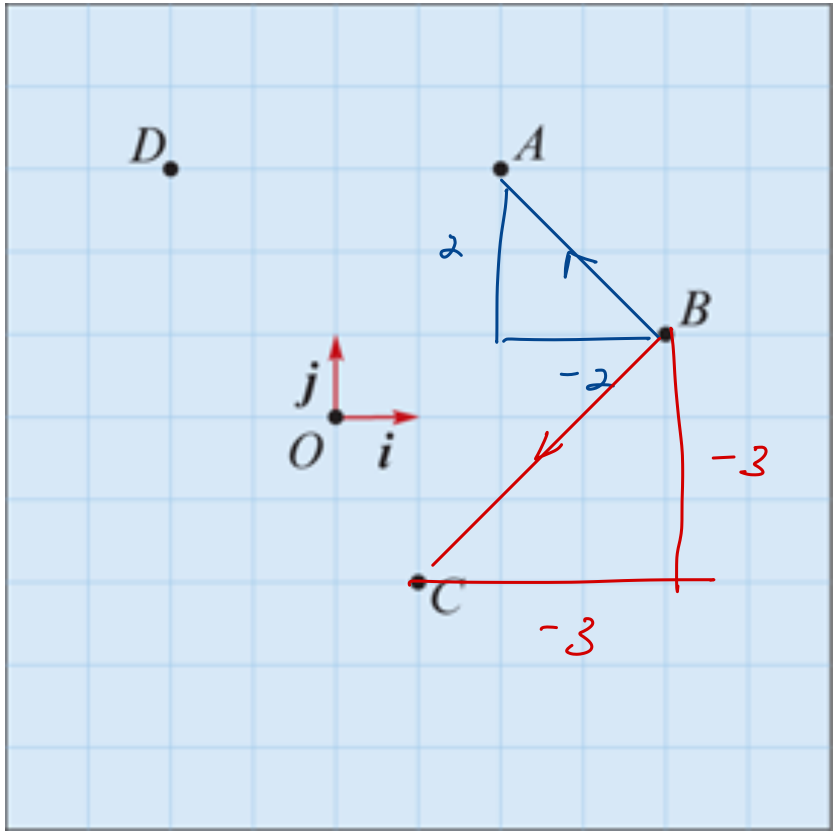
Find the magnitude of the vectors:

$\overrightarrow{AB}$

$\overrightarrow{BC}$

$$|\overrightarrow{AB}| = \sqrt{(-2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$|\overrightarrow{BC}| = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$$



Example

Let $a = i + 2j - k$, $b = 3i - 2k$ and $c = 2i + j + k$. Find:

$a + b$

$a - 2b$

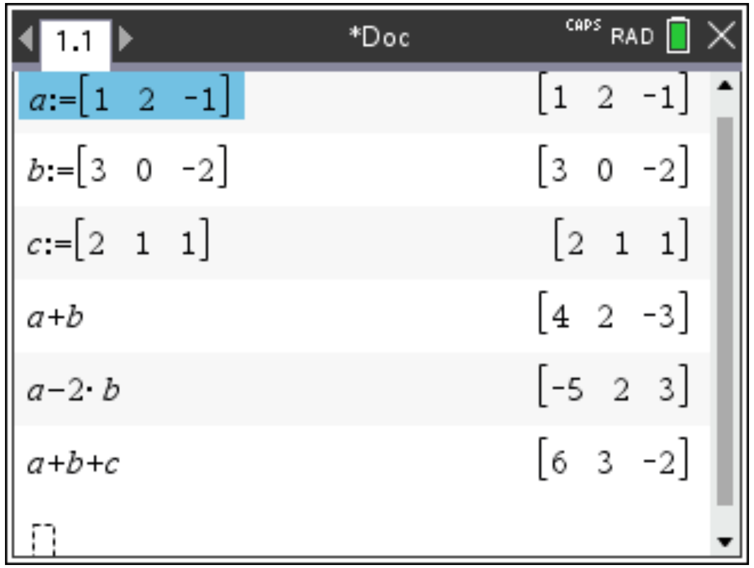
$a + b + c$

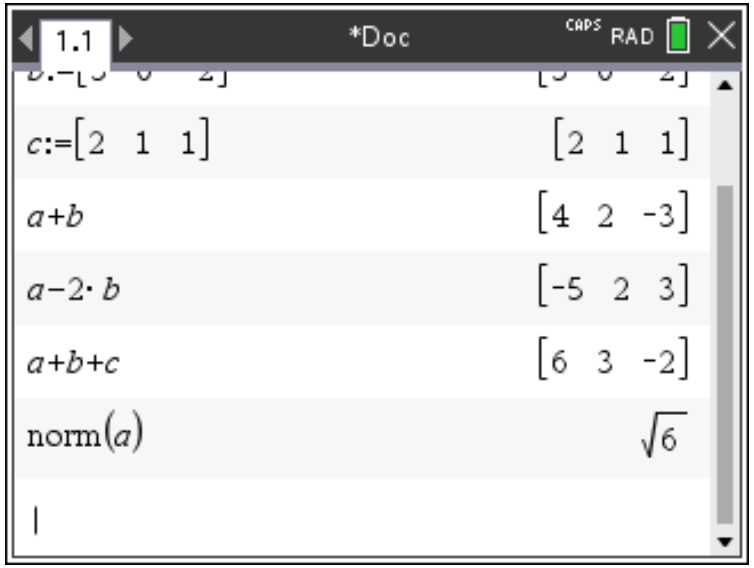
$|a|$

$4i + 2j - 3k$

$-5i + 2j + 3k$

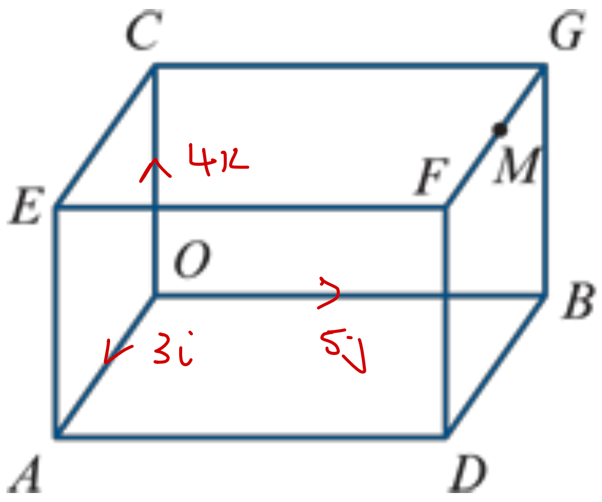
$6i + 3j - 2k$





Example

A cuboid is labelled as shown



$\overrightarrow{OA} = 3i, \overrightarrow{OB} = 5j$ and $\overrightarrow{OC} = 4k$

a. Find in terms of i, j and k:

i. $\overrightarrow{DB}$ $\quad -3i$

ii. $\overrightarrow{OD}$ $\quad 3i + 5j$

iii. $\overrightarrow{DF}$ $\quad 4k$

iv. $\overrightarrow{OF}$ $\quad \overrightarrow{OF} = 3i + 5j + 4k.$

b. Find $|\overrightarrow{OF}|$

$$|\overrightarrow{OF}| = \sqrt{3^2 + 5^2 + 4^2}$$
$$= \sqrt{9 + 25 + 16}$$
$$= \sqrt{50}$$
$$= 5\sqrt{2}$$

c. If M is the midpoint of FG, find:

i. $\overrightarrow{OM}$

ii. $|\overrightarrow{OM}|$

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AD} + \overrightarrow{DF} + \overrightarrow{FM}$$
$$= 3i + 5j + 4k + \frac{1}{2}\overrightarrow{FG}$$
$$= 3i + 5j + 4k + \frac{1}{2}(-3i)$$
$$= \frac{3}{2}i + 5j + 4k$$

$$|\overrightarrow{OM}| = \frac{1}{2}\sqrt{173}$$

Example

Let $A = (2, -4, 5)$ and $B = (5, 1, 7)$. Find M, the midpoint of AB.

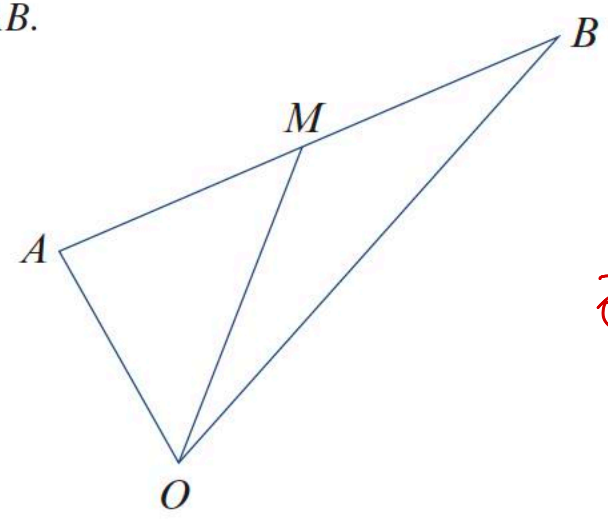
$$\text{Midpt} = \left(\frac{2+5}{2}, \frac{-4+1}{2}, \frac{5+7}{2}\right)$$

$$= \left(\frac{3}{2}, \frac{-3}{2}, 6\right)$$

Hint:
Always draw a diagram. Specialist Maths relies heavily on geometry. They are counting on most students trying to do this in their heads.

Example

Let $A = (2, -4, 5)$ and $B = (5, 1, 7)$. Find M, the midpoint of AB.



Hint:
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$$\vec{AB} = \vec{AO} + \vec{OB}$$

$$= -2i + 4j - 5k + 5i + j + 7k$$

$$= 3i + 5j + 2k$$

$$\vec{AM} = \frac{1}{2}\vec{AB} = \frac{3}{2}i + \frac{5}{2}j + k$$

$$\vec{OM} = OA + AM$$

$$= 2i - 4j + 5k + \frac{3}{2}i + \frac{5}{2}j + k$$

$$= \frac{7}{2}i - \frac{3}{2}j + 6k$$

Example with our old friend linear dependence and independence

We missed you old friend!

Example:

- Show that the vectors $a = 8i + 7j + 3k$, $b = i - j + 3k$ and $c = 2i + 3j - k$ are linearly dependent.
- Show that the vectors $a = 8i + 7j + 3k$, $b = i - j + 3k$ and $c = 2i + 3j + k$ are linearly independent.

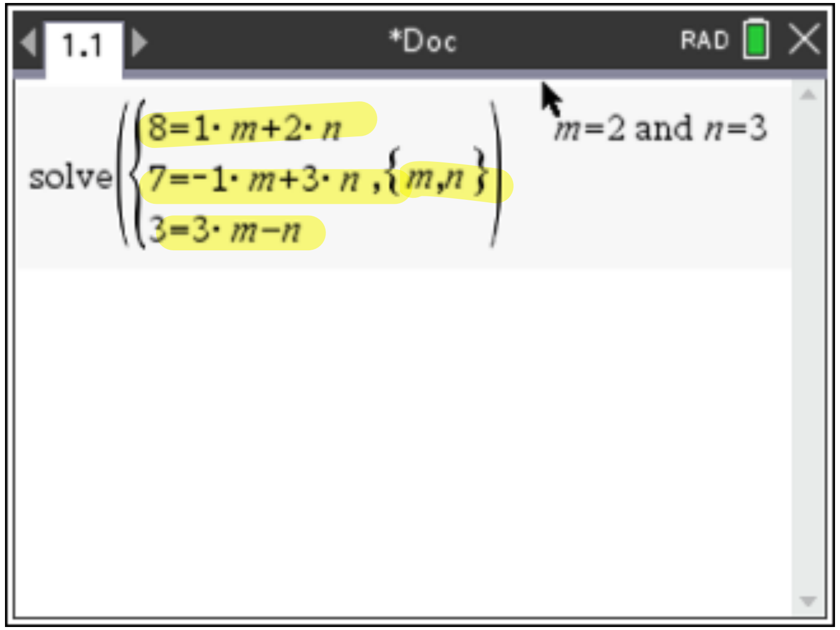
We can use vectors or just equate i, j and k terms ,,,

$$\begin{bmatrix} 8 \\ 7 \\ 3 \end{bmatrix} = m\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + n\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

$$8 = m + 2n$$

$$7 = -m + 3n$$

$$3 = 3m - n$$

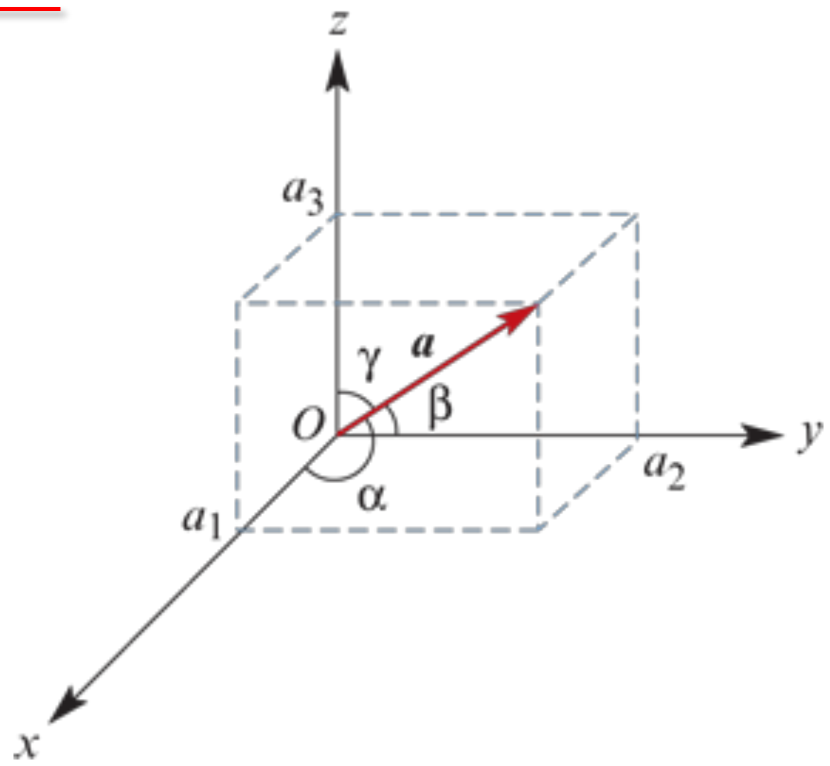


Angle made by a vector with an axis

The direction of a vector can be given by the angles which the vector makes with the i, j and k directions.

If we have a vector $a = a_1 i + a_2 j + a_3 k$ and we assume the angles made are α, β and γ respectively … then the angles made with the positive directions of the x-, y- and z-axes are:

$$\cos\alpha = \frac{a_1}{|a|}, \cos\beta = \frac{a_2}{|a|} \; and \; \cos\gamma = \frac{a_3}{|a|}$$



Looking at this over 2-dimensions make sense …

Examples

Let $a = 2i - j$ and $b = i + 4j - 3k$.

For each of these vectors, find:

- its magnitude
- the angle the vector makes with the y-axis.

$|a| = \sqrt{2^2 + (-1)^2}$

$= \sqrt{5}$

$|b| = \sqrt{1^2 + 4^2 + (-3)^2}$

$= \sqrt{1 + 16 + 9}$

$= \sqrt{26}$

$\cos\beta = \frac{-1}{\sqrt{3}} = 116{\cdot}57^\circ$

$\cos\beta = \frac{4}{\sqrt{26}} = 38{\cdot}33^\circ$

Examples

A position vector in two dimensions has magnitude 5 and its direction, measured anticlockwise from the x-axis, is 150°. Express this vector in terms of i and j.

With this question we can express the angle in two ways; with respect to the x-axis and the positive y-axis,

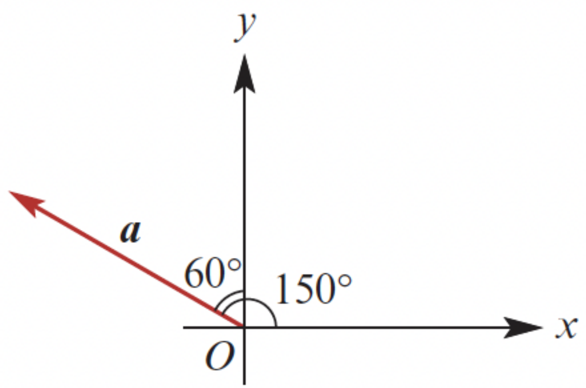
Using the general form $a = a_1 i + a_2 j$

$|a| = 5$

$\cos 150 = \frac{a_1}{5}$ $\cos 60 = \frac{a_2}{5}$

$a_1 = 5 \times \cos 150°$ $a_2 = 5 . \cos 60°$

$\therefore a = -\frac{5\sqrt{3}}{2} i + \frac{5}{2} j$



Again, a quick diagram makes it much easier.

Learning Objectives: Revisited

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Learning Objectives: Reviewed

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Work to be completed

The following represents the minimum work which should be completed.

The more questions you answer from each exercise, chapter review and Checkpoints the better you chance of gaining an excellent study score in November.

Specialist Mathematics Units 3 and 4 Textbook

Chapter 4
Exercise 4B: Resolution of a vector into rectangular components
Questions: 4, 7, 8, 10, 11, 15, 18, 19, 23, 24, 26, 28, 30, 31, 34, 35, 37