Introduction to vectors

Year 12 Specialist Maths
Units 3 and 4

www.maffsguru.com

Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what is meant be the following terms:
  - Directed line segments
  - Column vectors
  - Vector notation
- Understand how to
  - Find the magnitude of a vector
  - Add vectors (geometrically and using column vectors)
  - Subtract vectors
- Understand what is meant by
  - scalar multiplication
  - The zero vector
  - Polygon of vectors
  - Parallel vectors
  - Three dimensional vectors
  - Linear dependence and independence

Recap

This is a new section of the Specialist Mathematics course and one which is pretty awesome! It builds on much of the work we have covered in other aspects of the VCE courses but now takes it a little deeper and wider.

The important learning from this section is about the language!

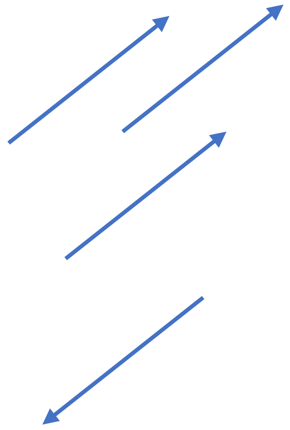
What is a vector?

A vector is a quantity which has a **direction** and a **magnitude**.

The direction is shown using an arrow and the magnitude is shown by the length of a line (using a suitable unit)

Lines having the same length and pointing in exactly the same direction are said to be equivalent.

The lines with arrows are called **directed line segments** and the sets of equivalent segments are called vectors.



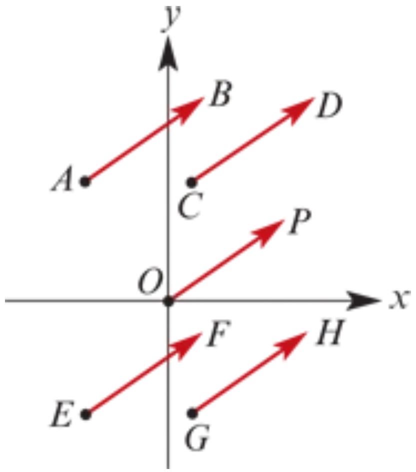
Directed line segments

Notation is really important!

The notation shown below is a directed line segment from A to B.

$$\overrightarrow{AB}$$

We can draw a graphical representation of this as shown below:



It's important to note that the line segments are all the same magnitude and pointing in the same direction.

Column vectors

We can express vectors in a number of different ways. Another helpful way is to use a column vector where the top number expresses the horizontal movement and the bottom number the vertical movement.

$$\overrightarrow{AB} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$\overrightarrow{AB} = 3i + 2j$

Negative and positive numbers denote a movement to the left/right and up/down respectively.

At the moment, we're going to think of vectors in reference to the cartesian plane.

Vector notation

In textbooks it's easier to use a lower case bold letter to stand for a vector. When handwriting we use a squiggly line under the letter as it's not easy to make something appear bold!

$$\boldsymbol{v} = \overrightarrow{AB}$$

An example of how we would handwrite the above is shown below:

$\underset{\sim}{v}$

The size or magnitude of a vector

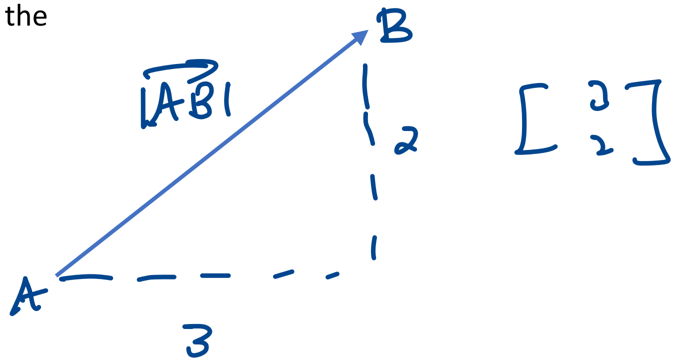
$|a|^2$

It’s going to be important to find the size (or magnitude) of a vector. **This is going to be required in a number of sections of the course**.

Due to a vector being a line between two points, we can use Pythagoras’ Theorem to find the length of the line (**magnitude**)

The magnitude is denoted in the following way:

$$|\overrightarrow{AB}|$$



Hence we could find the magnitude of the following vector quite easily!

$\overrightarrow{AB}=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

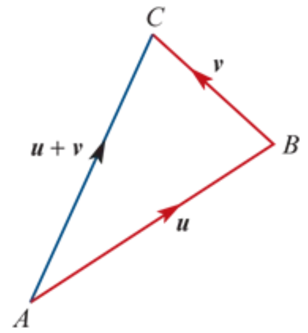
$$|\overrightarrow{AB}| = \sqrt{3^2 + 2^2}$$

$$= \sqrt{9 + 4} = \sqrt{13}$$

Adding vectors (using a diagram)

I like to think of vectors as one way streets. We travel along them from a start point to an end point. We probably don't always take the most direct route. Hence, when adding vectors together, we're looking at the journey we could have taken if we'd been able to fly as the crow flies (directly from the start to the end).

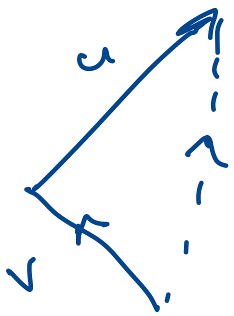
For example:



$u + v$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

Note: The sum has two geometric representations



Adding vectors (using column vectors)

Adding vectors together when they are in column form makes the processes somewhat trivial.

For example, we can add the following two column vectors together:

$$u = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \; and \; v = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Adding them together gives:

$$u + v = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

This is the same as representing the vectors geometrically as:

Affecting the length of the vectors using multiplication

We can alter the length of a vector using scalar multiplication. A scalar is a number with **magnitude** but no direction.

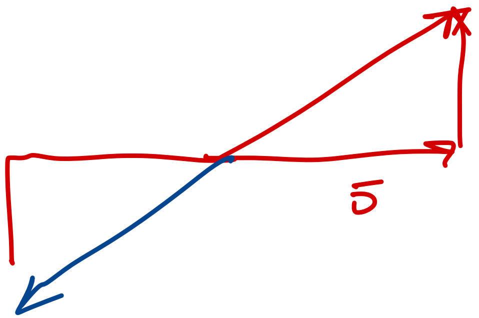
Hence, if we have the following vector

$$u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

The vector $2u$ represents a line segment heading in the **same direction** as u but **twice** as long.

Note: We can multiply by a negative number which will change the direction of the line (by rotating it 180°)

Note: $-\overrightarrow{AB} = \overrightarrow{BA}$



$$\underset{\sim}{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$2\underset{\sim}{u} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$$

$$-\underset{\sim}{u} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$

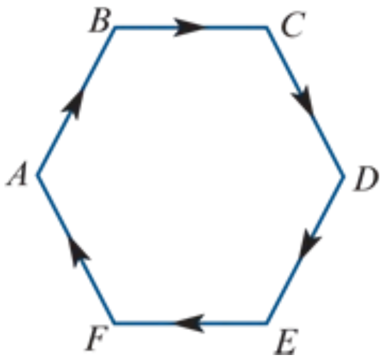
A line segment with no length?

I can honestly say that I think this is a little pointless, but here we go!

When we have a line segment of zero length it is called the **zero vector.** It also has no direction (which seems an obvious fact!).

To take the whole "stating the obvious" one step further, it also has no magnitude.

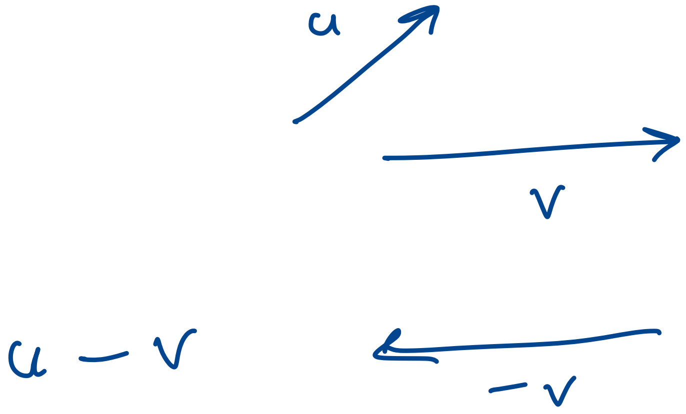
As an example, if we were to add the following vectors, we would have the zero vector as we end where we start.



We can also subtract vectors

It’s important to note that we can also subtract vectors.

When we do this, it’s nice to view it geometrically. If we were to have the following, we can present the subtraction of vectors u and v as follows:



Examples

Draw a directed line segment representing the vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and state the magnitude of this vector.

$$|\underset{\sim}{u}| = \sqrt{3^2 + (-2)^2}$$

$$= \sqrt{9 + 4}$$

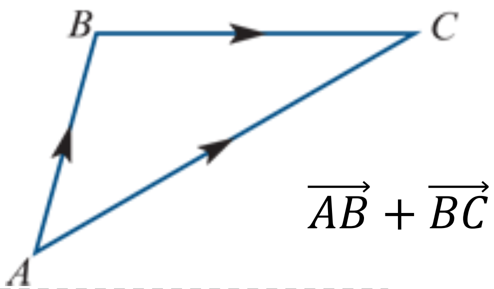
$$= \sqrt{13}$$

Polygon of vectors

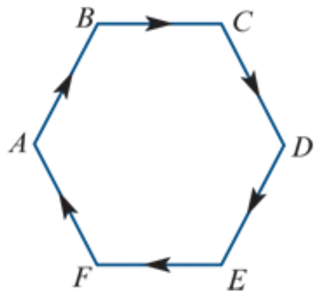
We know that polygons are defined as:

"A plane figure that is described by a finite number of straight line segments connected to form a closed polygonal chain or polygonal circuit"

Two examples are shown below:



$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \mathbf{0}$$

Parallel vectors

As we know that parallel lines will never ever meet, it would hence support the idea that parallel vectors will have the same direction (but not necessarily the same magnitude)

As direction is given as a column vector, we can compare vectors to see if they are parallel.

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \; and \; v = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

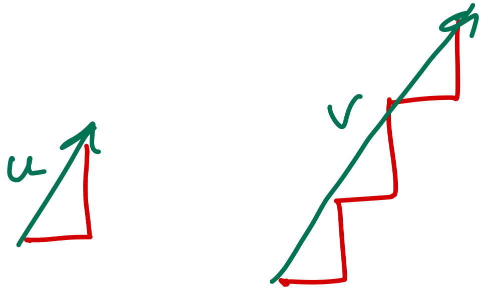
$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\quad v = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$

These are parallel as v is simply 3 times longer than u.

$v = 3u = 3\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Important note: Two non-zero vectors are parallel if there is some $k \in \mathbb{R}$ such that $u = kv$

The above rule comes in really important in later exercises



Position vectors

When vectors are drawn relative to the origin they are known as position vectors.

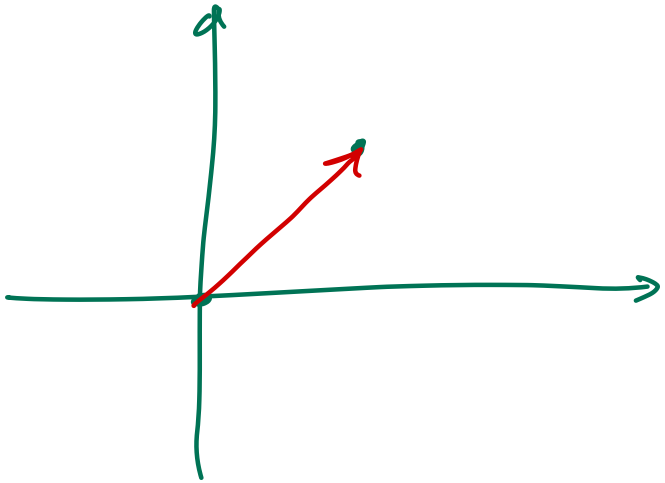
Note: None of the vectors we have looked at so far are position vectors.

We denote a position vector by starting with the letter “0” (not zero!)

$\overrightarrow{OA}$

Position vectors come in handy later when we need to prove things

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

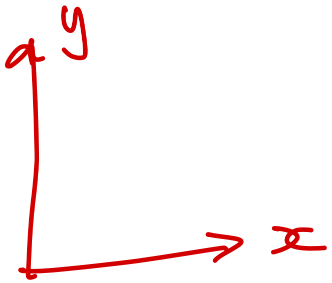


Three dimensional vectors

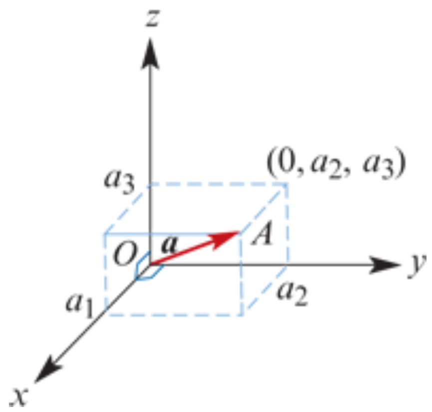
We all live in three dimensions and so can vectors!

It's important to note the convention when using vectors in three dimensions.

Two dimensional vectors have the x and y axes as we expect.



Three dimensional vectors have them as follows with the x coming out of the page:



Common sense rules

This course is all about the properties!!

Here are some common sense properties for vectors:

commutative law for vector addition	$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$
associative law for vector addition	$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$
zero vector	$\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}$
additive inverse	$\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$
distributive law	$m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$, for $m \in \mathbb{R}$

Now for something completely different

It’s going to come up time and again that you need to understand the difference between linear dependence and independence.

First the definition:

A set of vectors is said to be **linearly dependent** if at least one of its members can be expressed as a linear combination of other vectors in the set.

A set of vectors is said to be **linearly independent** if it is not linearly dependent. That is, a set of vectors is linearly independent if no vector in the set is expressible as a linear combination of other vectors in the set.

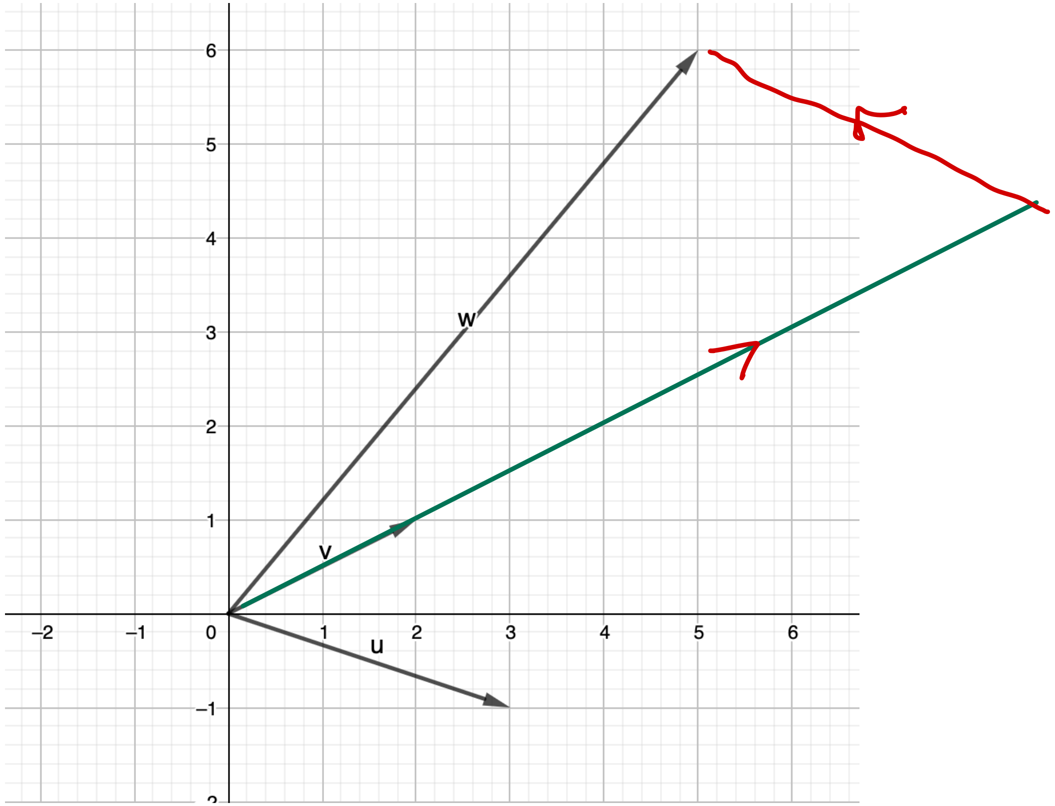
So if something is **linearly dependent**, you can take two vectors and change their scale and, when you add them together you get the third.

Graphically speaking

Let's look at the vectors given below and plot them on a set of axes

$$u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$v = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$
$$w = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

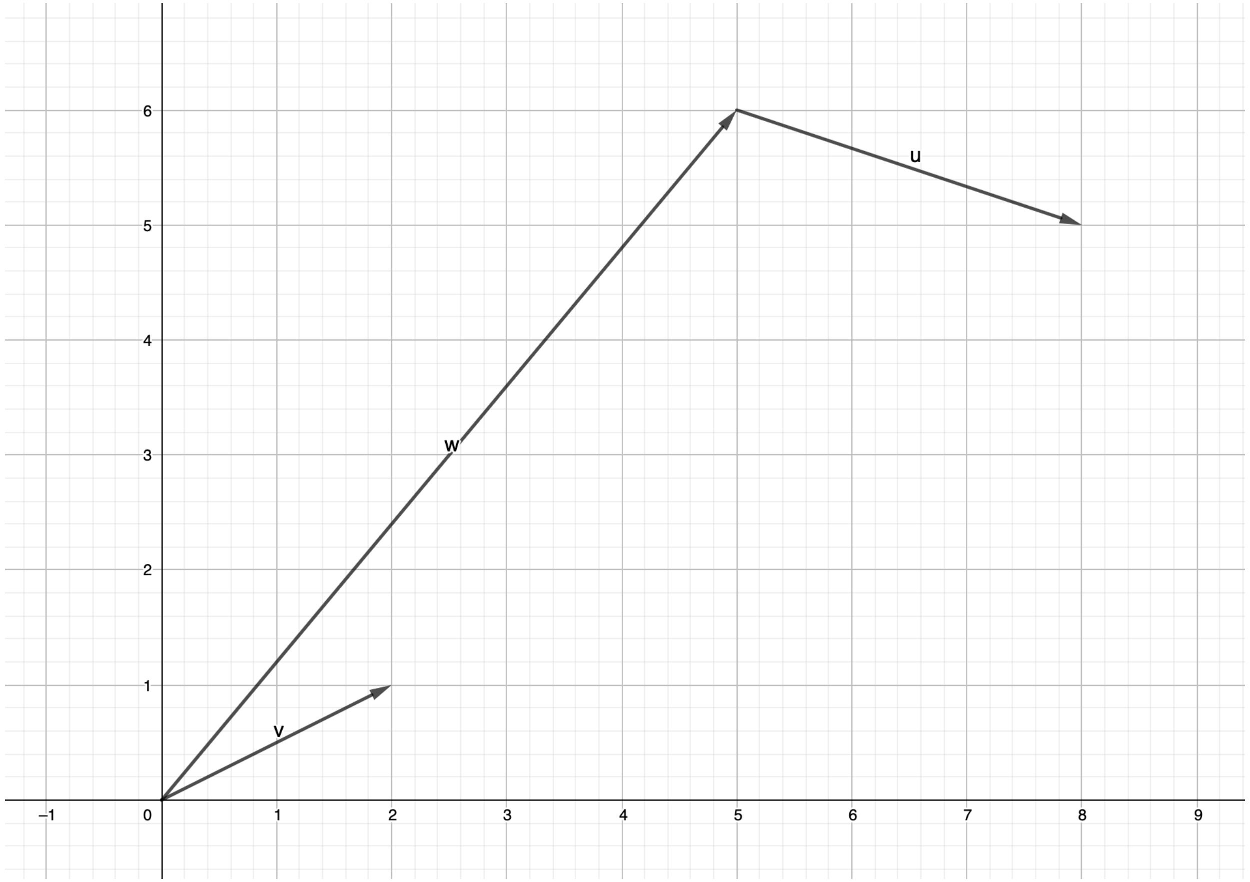
Can I create one from a combination of the other two?



Graphically speaking

Joining them together we can see something interesting. If we extend u and v by some factor *m* and *n* (for example). And, in this case reverse the vector u, then we can create w.

If we can take three vectors and, by multiplying two of them to create the third, then we say the vectors are **linearly dependent**.



Important information

It's important to note the following:

A vector is linearly dependent if it can be expressed in the form:

$$w = k_1 v_1 + k_2 v_2 + k_3 v_3$$

Where the values of k are real numbers.

When there are two vectors, they are only linearly dependent if and only if they are parallel.

A set of three or more vectors is linearly dependent if and only if there exist real numbers a, b and c such that:

$$au + bv + cw = \mathbf{0}$$

Any set that contains the zero vector is linearly dependent.
Any set of three or more two-dimensional vectors is linearly dependent.
Any set of four or more three-dimensional vectors is linearly dependent

Important information

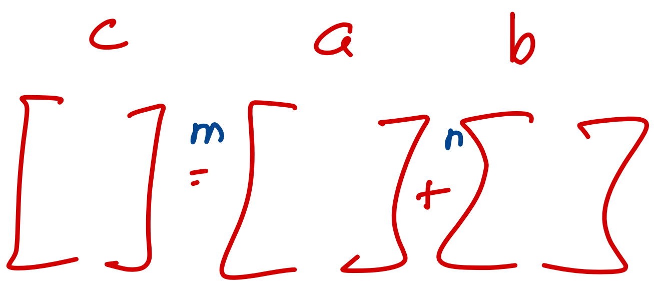
We are going to need to use the following for three vectors to show if they are linearly dependent or not.

$$c = ma + nb$$

Where vector c is a combination of vectors a and b:

It's also important to note that when two vectors are linearly independent then the following implies that $m = p \text{ and } n = q$:

$$ma + nb = pa + qb$$



$c = ma + nb$

Examples

Determine whether the following sets of vectors are linearly dependent:

$$a = \begin{bmatrix}2\\1\end{bmatrix}, b = \begin{bmatrix}3\\-1\end{bmatrix}\ and\ c = \begin{bmatrix}5\\6\end{bmatrix}$$

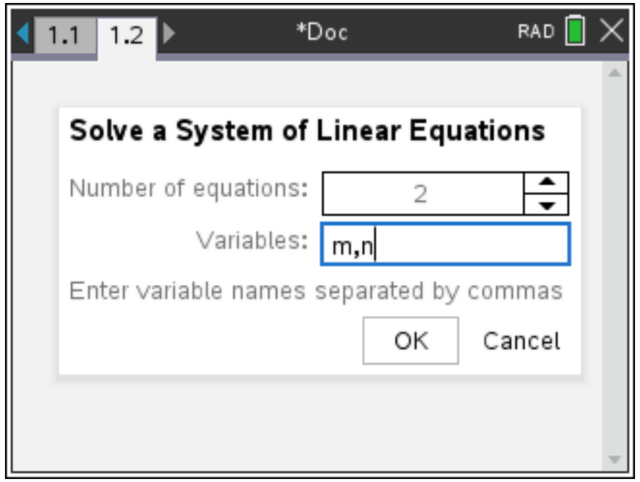
Note: It's important to check if any of the vectors are parallel. If they are, they need to be on opposite sides of the equals sign.

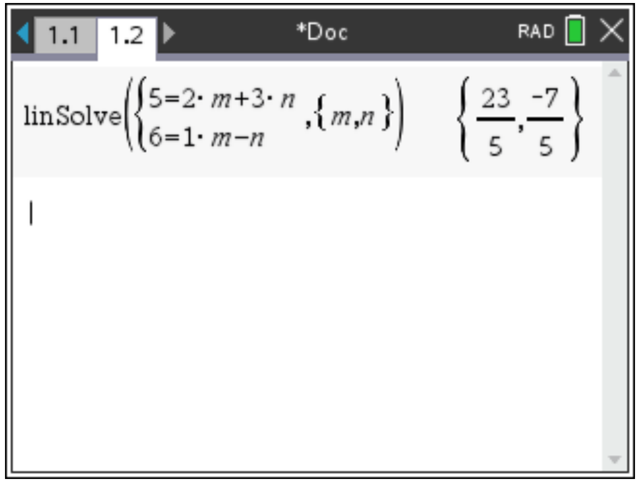
As none of them are parallel …we can use $c = ma + nb$

$$\begin{bmatrix}5\\6\end{bmatrix} = m\begin{bmatrix}2\\3\end{bmatrix} + n\begin{bmatrix}3\\-1\end{bmatrix}$$

$$5 = 2m + 3n$$

$$6 = 3m - n$$





As there is a solution, then we know the vectors are linearly dependent

Examples

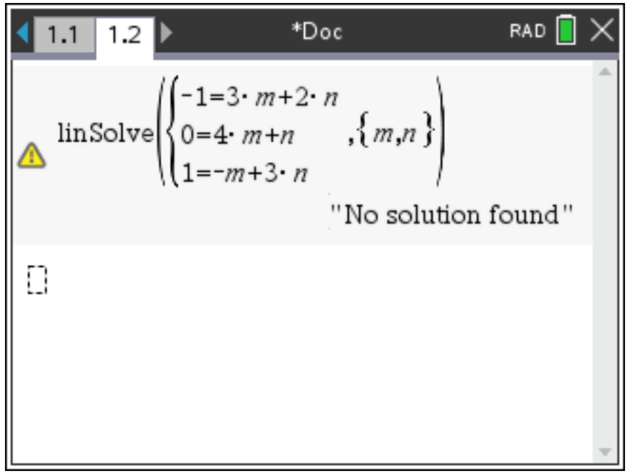
Determine whether the following sets of vectors are linearly dependent:

$$a = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \text{ and } c = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Note: It's important to check if any of the vectors are parallel. If they are, they need to be on opposite sides of the equals sign.

As none of them are parallel …we can use $c = ma + nb$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = m\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} + n\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$



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Examples

Points A and B have position vectors a and b respectively, relative to an origin O.

The point D is such that $\overrightarrow{OD} = k\overrightarrow{OA}$ and the point E is such that $\overrightarrow{AE} = l\overrightarrow{AB}$. The line segments BD and OE intersect at X.

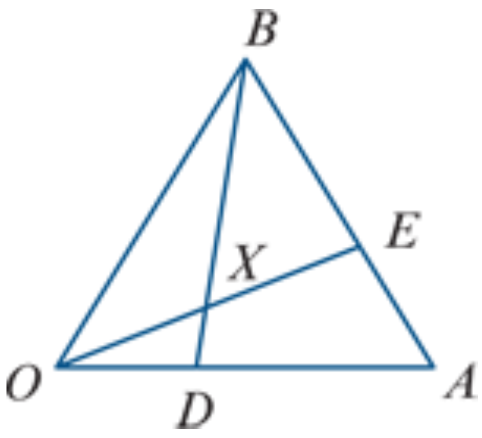
Assume that $\overrightarrow{OX} = \frac{2}{5}\overrightarrow{OE}$ and $\overrightarrow{XB} = \frac{4}{5}\overrightarrow{DB}$.

Express $\overrightarrow{XB}$ in terms of a, b and k.

Express $\overrightarrow{OX}$ in terms of a, b and l.

Express $\overrightarrow{XB}$ in terms of a, b and l.

Find k and l.



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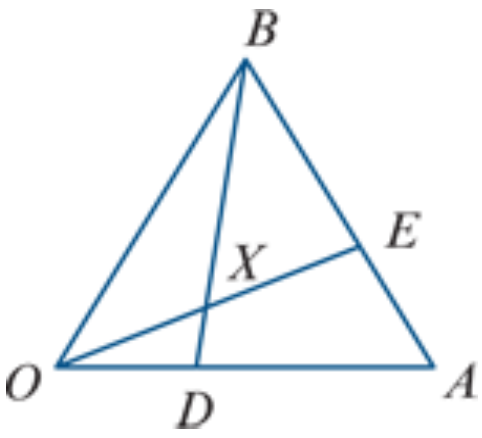

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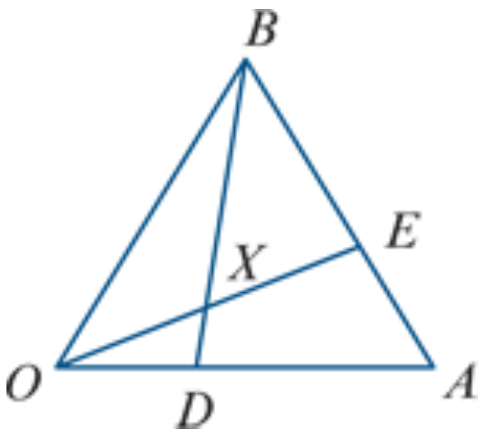

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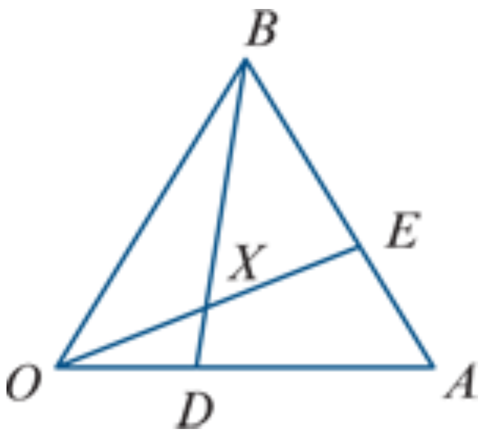


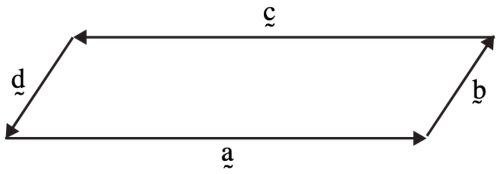
Note: It's important to note that we have been asked to find $\overrightarrow{XB}$ in terms of different variables. As the vector is the same and they are linearly independent, we can equate the two vectors and compare coefficients of a and b

VCAA Questions

Specialist Maths
2006 Paper 2

Question 15

In the parallelogram shown, $|\underset{\sim}{a}| = 2|\underset{\sim}{b}|$.



Which one of the following statements is true?

A. $\underset{\sim}{a} = 2\underset{\sim}{b}$

B. $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$

C. $\underset{\sim}{b} - \underset{\sim}{d} = \underset{\sim}{0}$

D. $\underset{\sim}{a} + \underset{\sim}{c} = \underset{\sim}{0}$

E. $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{c} - \underset{\sim}{d}$

Question 13

The vectors $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = \lambda\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{k}$ will be **linearly dependent** when the value of λ is

A. 1

B. 2

C. 3

D. 4

E. 5

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = m\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + n\begin{bmatrix} \lambda \\ 3 \\ 2 \end{bmatrix}$$

Specialist Maths
2018 Paper 2

Question 12

If $|\underset{\sim}{a}+\underset{\sim}{b}| = |\underset{\sim}{a}| + |\underset{\sim}{b}|$ and $\underset{\sim}{a}, \underset{\sim}{b} \neq \underset{\sim}{0}$, which one of the following is **necessarily true**?

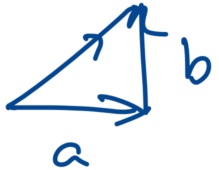
A. $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$

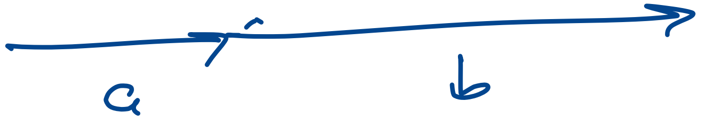
B. $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

C. $\underset{\sim}{a} = \underset{\sim}{b}$

D. $\underset{\sim}{a} = -\underset{\sim}{b}$

E. $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$





VCAA Questions

Specialist Maths
2018 Paper 2

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B. $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

C. $\underset{\sim}{a} = \underset{\sim}{b}$

D. $\underset{\sim}{a} = -\underset{\sim}{b}$

E. $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$

Option B would not necessarily satisfy the given statement. Options D and E would not satisfy the given statement.

Options A and C would satisfy the given statement, but only A is necessarily true.

Properties of the modulus function

- $|ab| = |a|\,|b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- $|x| = a$ implies $x = a$ or $x = -a$
- $|a + b| \leq |a| + |b|$
- If a and b are both positive or both negative, then $|a + b| = |a| + |b|$.
- If $a \geq 0$, then $|x| \leq a$ is equivalent to $-a \leq x \leq a$.
- If $a \geq 0$, then $|x - k| \leq a$ is equivalent to $k - a \leq x \leq k + a$.

Work to be completed

The following represents the minimum work which should be completed.

The more questions you answer from each exercise, chapter review and Checkpoints the better you chance of gaining an excellent study score in November.

Specialist Mathematics Units 3 and 4 Textbook

Chapter 2
Exercise 2A: Introduction to Vectors
Questions: 3, 4, 5, 7, 8, 10, 12, 14, 15, 16, 18