The reciprocal circular functions

Year 12 Specialist Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what the cosecant function is and what it looks like
- Understand what the secant function is and what it looks like
- Understand what the cotangent function is and what it looks like
- Understand the useful properties of each of the above
- Understand how to use the CAS to solve functions containing the above.

Recap

We have, in previous sections of this course, looked at how we can use the sec function with parametric equations. Namely:

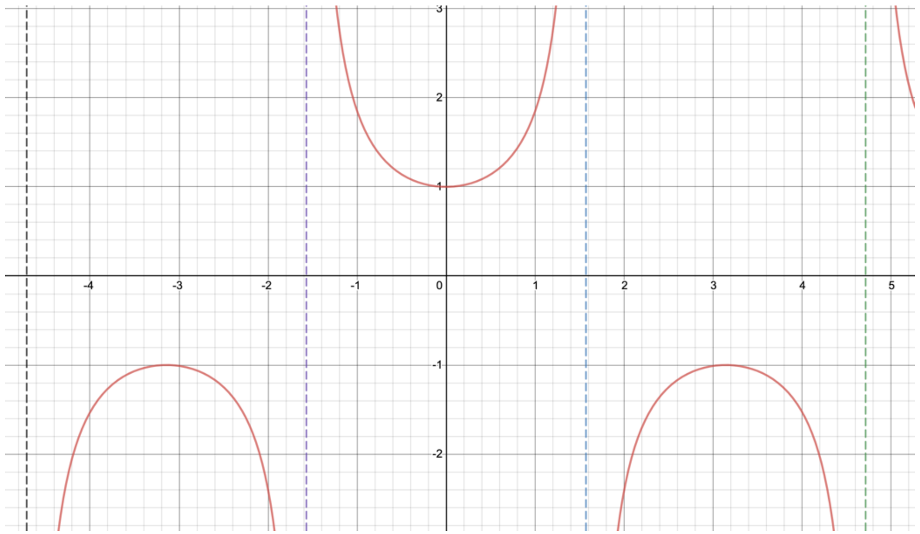
$$x = a\sec t$$
$$y = b\tan t$$

And

$$\sec^2\theta - \tan^2\theta = 1$$

To convert from parametric form to cartesian form for hyperbolas.

It's also important to look at the form of the graph to allow us to find the domain and range of cartesian equations. The sec function was particular interesting.



The graph of $\sin\theta$ and $cosec\theta$

We know what the graph of $\sin\theta$ looks like! We can use the graph to gain an understanding of what $cosec\ \theta$ would look like. Remember:

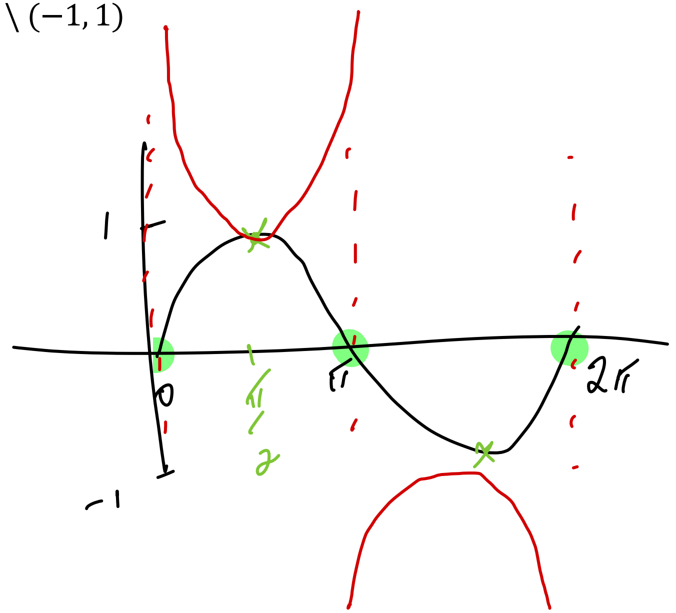
$$cosec\ \theta = \frac{1}{\sin\theta}$$

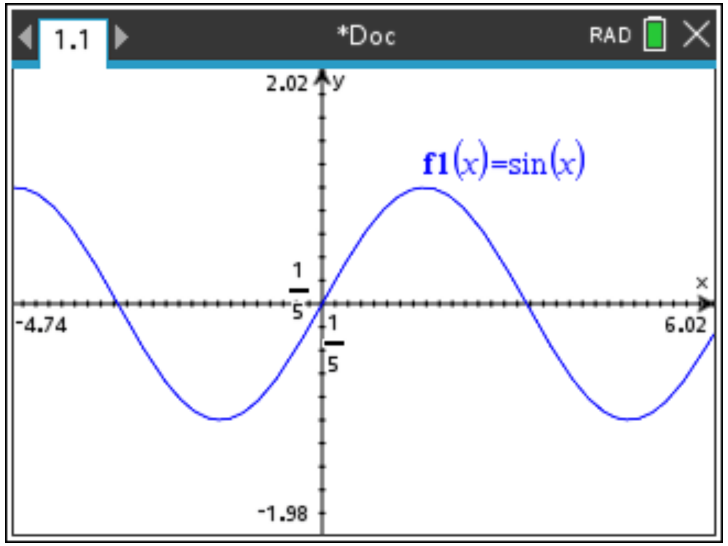
In the above equation, we know that the value of $\sin\theta$ cannot be zero. Hence we know there are asymptotes!

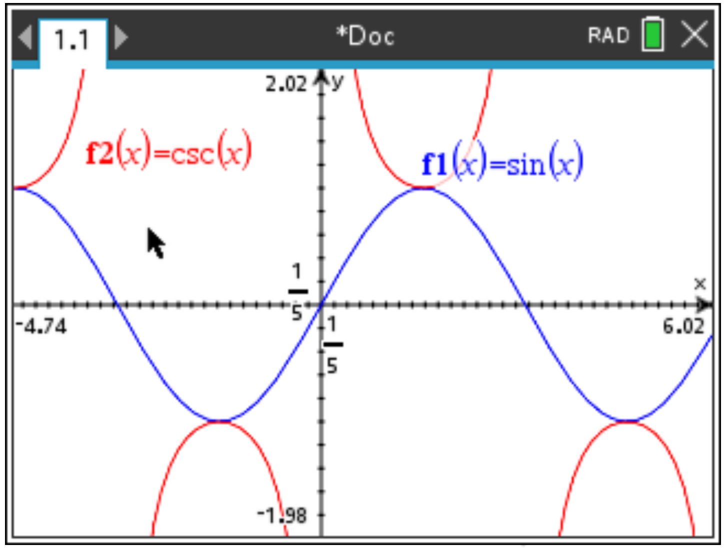
These are at $\theta = n\pi, for\ n \in \mathbb{Z}$

Hence, the domain for $cosec\theta$ is $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$. The turning points are the same as they are for $\sin\theta$

Range: We can see the range is $\mathbb{R} \setminus (-1, 1)$







The graph of $\cos\theta$ and $\sec\theta$

$2n$, $n \in \mathbb{Z}$

$2n+1$, $n \in \mathbb{Z}$

We can use the same idea with

$\frac{1\pi}{2}$ $\frac{3\pi}{2}$

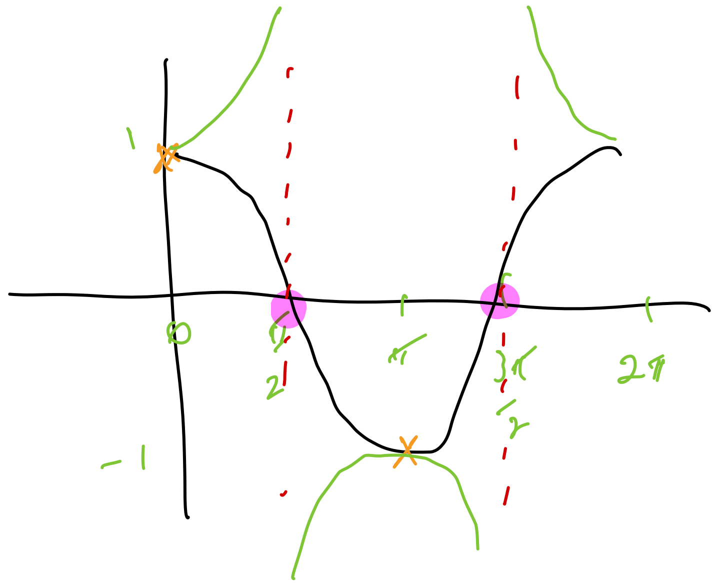
$$\sec\theta = \frac{1}{\cos\theta}$$

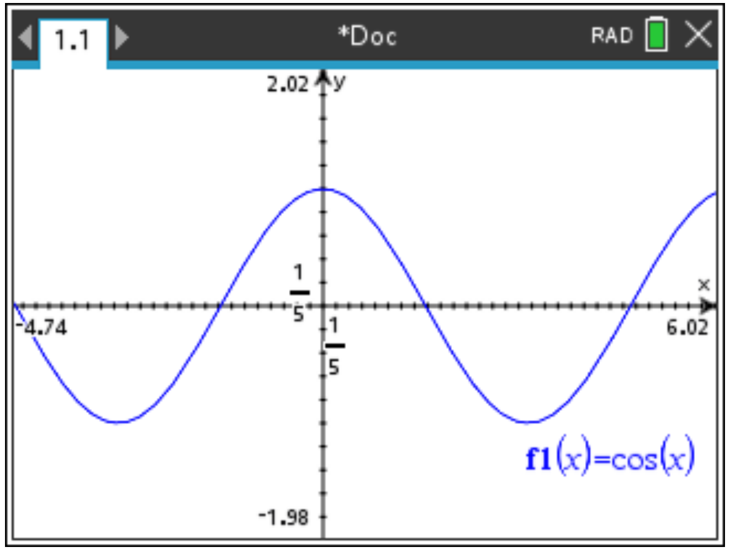
We must know that the value of $\cos\theta$ cannot be zero. Hence we know there are asymptotes!

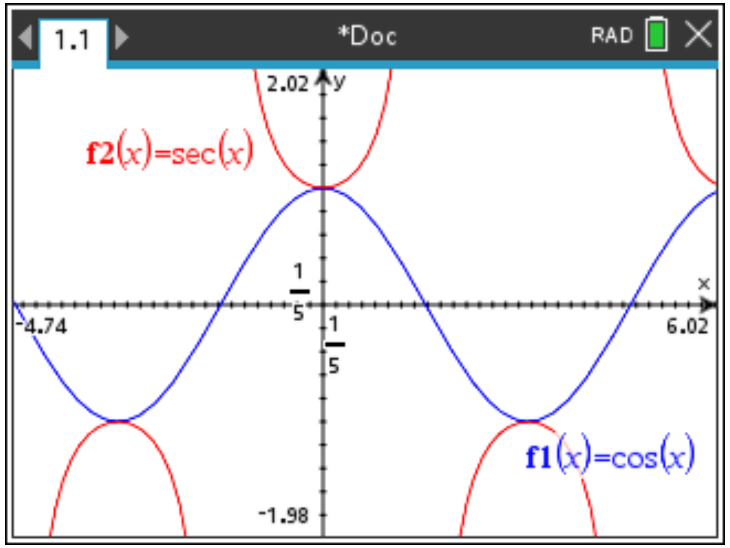
These are at $\theta = \frac{(2n+1)\pi}{2}, for\ n \in \mathbb{Z}$

Hence, the domain is $\mathbb{R} \setminus \{\frac{(2n+1)\pi}{2}, for\ n \in \mathbb{Z}\}$

The turning points are the same as they are for $\cos\theta$. Range: We can see the range is $\mathbb{R} \setminus (-1, 1)$







The graph of $tan\ \theta$ and $cot\ \theta$

We know from past theory that we can express $\cot\theta$ as

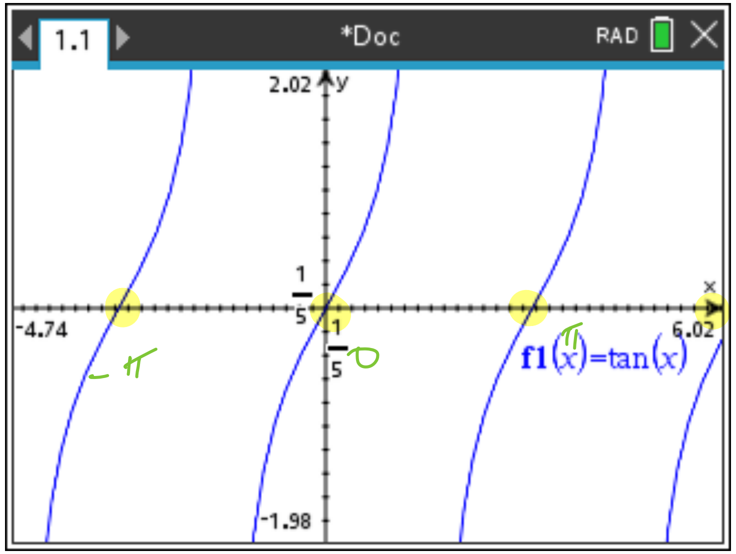
$$cot\ \theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{tan\theta}$$

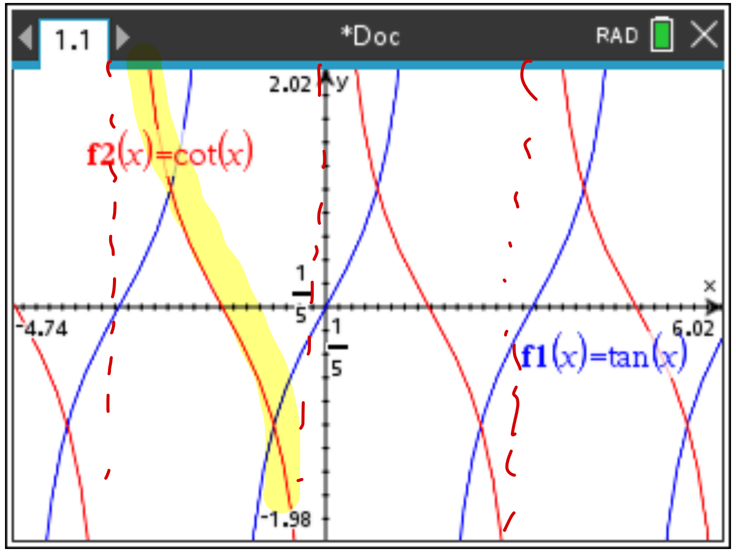
We can prove (and perhaps see) that the graph of $y = \cot\theta$ is obtained from the graph of $y = \tan\theta$ with a translation of $\frac{\pi}{2}$ to the left and a reflection in the $x - axis$.

This gives the following as true:

Domain is $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$. Range is $\mathbb{R}$

Asymptotes are found at $\theta = n\pi, for\ n \in \mathbb{Z}$



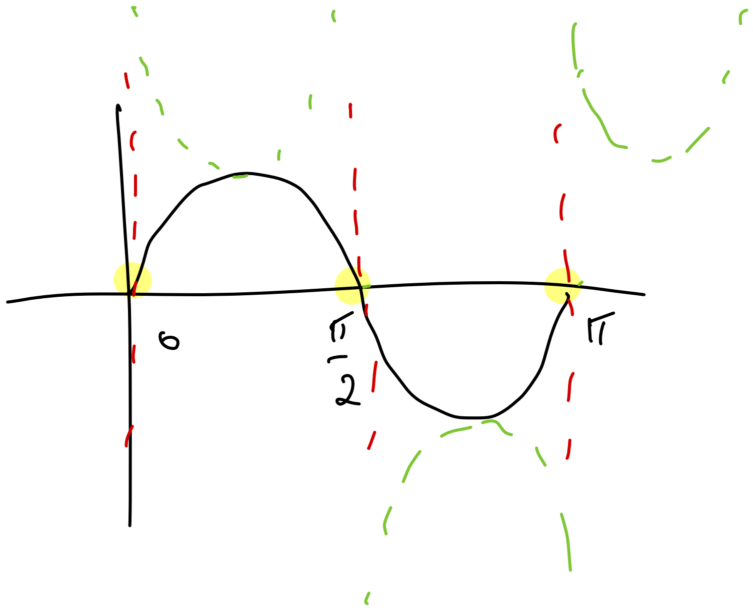


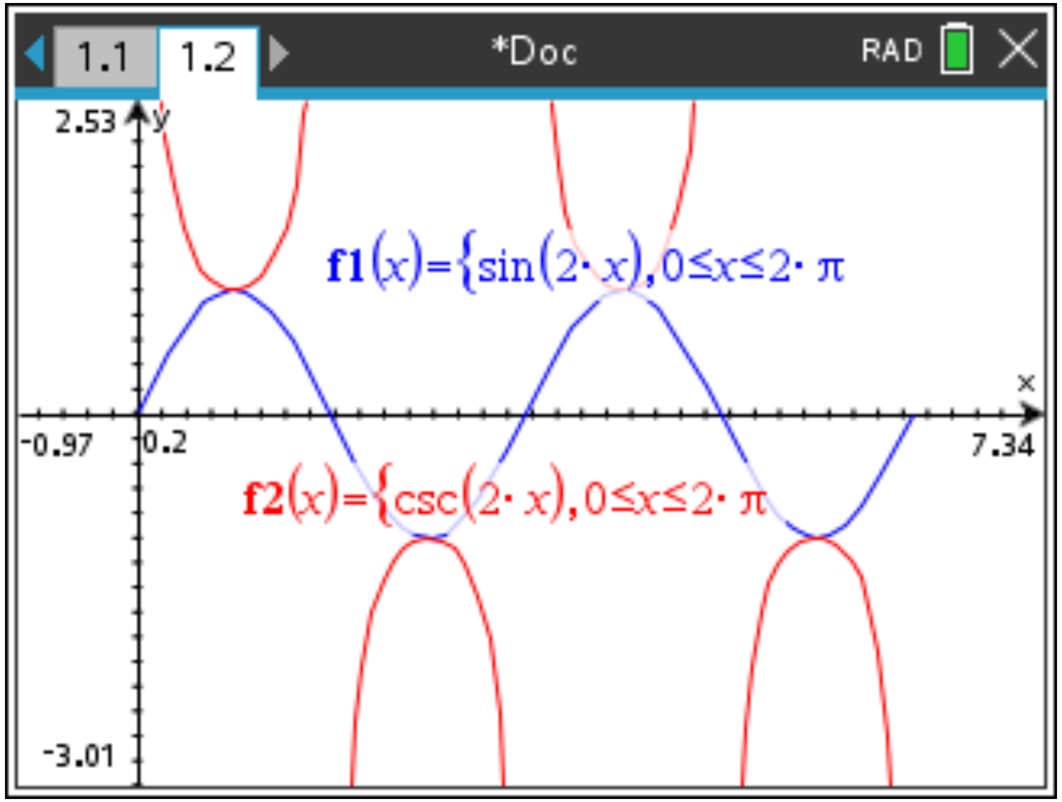
Examples

Sketch the graph of the following over the interval $[0,2\pi]$:

$y = cosec\ (2x)$

It's always helpful to refer to the "base" graph of the cosec function then use DRT as we would have done in Methods 1 and 2 (and Methods 3 and 4).





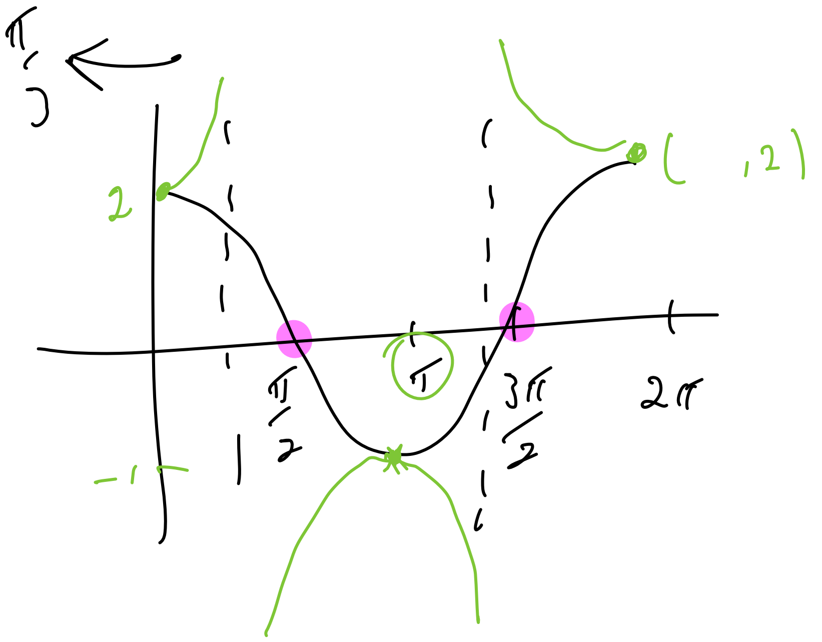
Examples

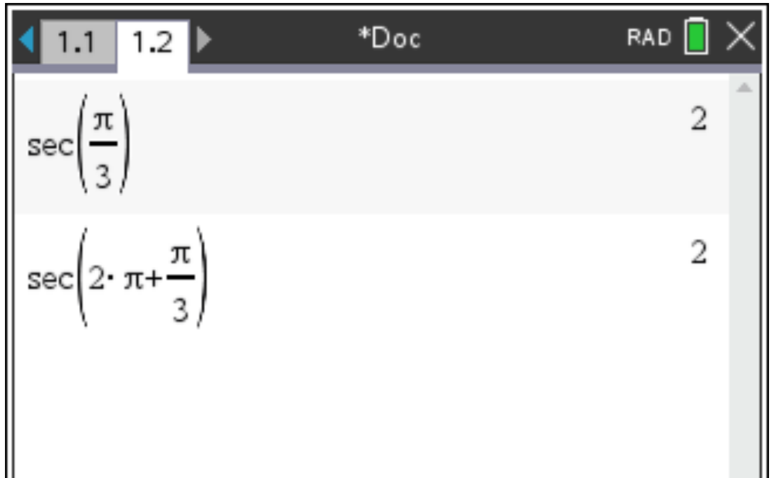
Sketch the graph of the following over the interval $[0,2\pi]$:

$y = sec\left(x + \frac{\pi}{3}\right)$

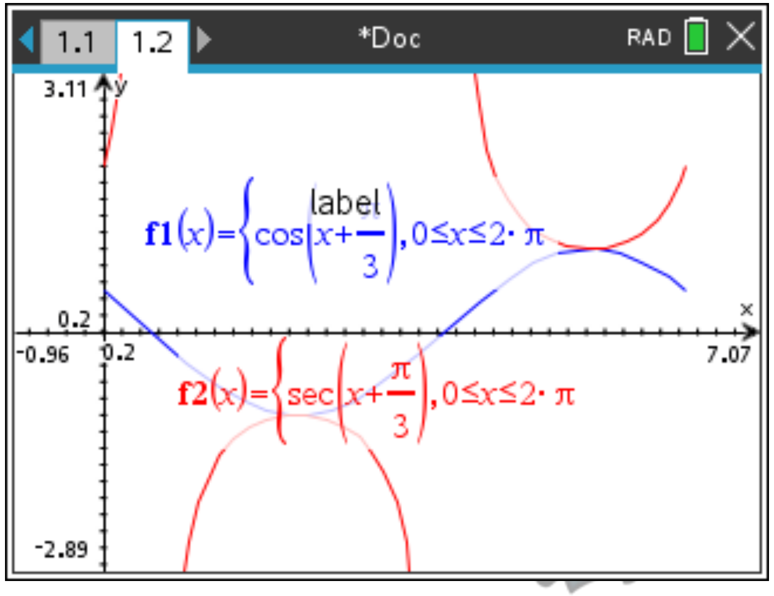
Once again, we can use the idea of the base function to help us translate it $\frac{\pi}{3}$ units to the left.

Remember to calculate the y-axis intercepts and limits of the function!





Use the CAS to help with end points

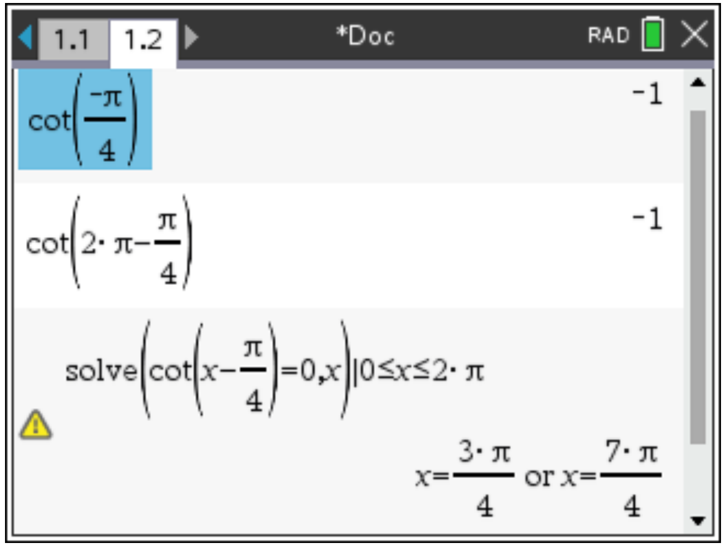


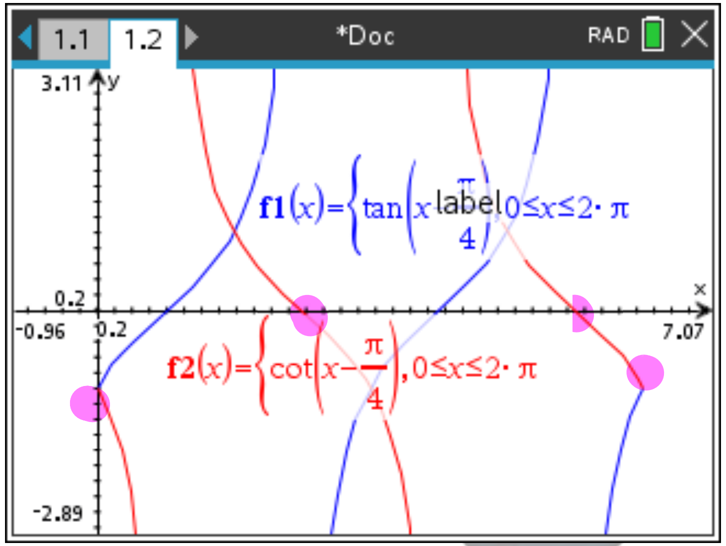
Examples

Sketch the graph of the following over the interval $[0,2\pi]$:

$y = cot\left(x - \frac{\pi}{4}\right)$

Once again, use the base graph to assist with drawing the above.



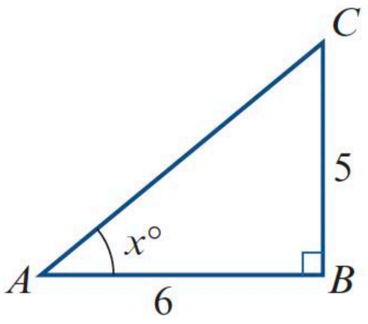


Examples

In triangle ABC, $\angle ABC = 90°$, $\angle CAB = x°$, $AB = 6$ cm and $BC = 5$ cm. Find:

a AC

b the trigonometric ratios related to $x°$



a. $AC^2 = 5^2 + 6^2$

$AC^2 = 25 + 36$

$AC^2 = 61$

$\therefore AC = \sqrt{61}$ cm

$\sin x = \dfrac{5}{\sqrt{61}}$ $\quad$ $\operatorname{cosec} x° = \dfrac{\sqrt{61}}{5}$

$\cos x = \dfrac{6}{\sqrt{61}}$ $\quad$ $\sec x° = \dfrac{\sqrt{61}}{6}$

$\tan x = \dfrac{5}{6}$ $\quad$ $\cot x° = \dfrac{6}{5}$

Useful properties

Much like we have used in Methods 1 and 2 (and Methods 3 and 4) we will find the following symmetry properties will come in very handy!

$$\begin{aligned} \sec(\pi - x) &= -\sec x \\ \sec(\pi + x) &= -\sec x \\ \sec(2\pi - x) &= \sec x \\ \sec(-x) &= \sec x \end{aligned} \qquad \begin{aligned} \operatorname{cosec}(\pi - x) &= \operatorname{cosec} x \\ \operatorname{cosec}(\pi + x) &= -\operatorname{cosec} x \\ \operatorname{cosec}(2\pi - x) &= -\operatorname{cosec} x \\ \operatorname{cosec}(-x) &= -\operatorname{cosec} x \end{aligned} \qquad \begin{aligned} \cot(\pi - x) &= -\cot x \\ \cot(\pi + x) &= \cot x \\ \cot(2\pi - x) &= -\cot x \\ \cot(-x) &= -\cot x \end{aligned}$$

$$\begin{aligned} \sec\left(\frac{\pi}{2} - x\right) &= \operatorname{cosec} x \\ \cot\left(\frac{\pi}{2} - x\right) &= \tan x \end{aligned} \qquad \begin{aligned} \operatorname{cosec}\left(\frac{\pi}{2} - x\right) &= \sec x \\ \tan\left(\frac{\pi}{2} - x\right) &= \cot x \end{aligned}$$

Examples using the properties

Find the exact value of each of the following:

$\mathbf{sec}\left(\frac{\mathbf{11\pi}}{\mathbf{4}}\right)$

$cosec\left(-\frac{23\pi}{4}\right)$

$\cot\left(\frac{11\pi}{3}\right)$

$$\begin{aligned}
\sec(\pi - x) &= -\sec x & \operatorname{cosec}(\pi - x) &= \operatorname{cosec} x & \cot(\pi - x) &= -\cot x \\
\sec(\pi + x) &= -\sec x & \operatorname{cosec}(\pi + x) &= -\operatorname{cosec} x & \cot(\pi + x) &= \cot x \\
\sec(2\pi - x) &= \sec x & \operatorname{cosec}(2\pi - x) &= -\operatorname{cosec} x & \cot(2\pi - x) &= -\cot x \\
\sec(-x) &= \sec x & \operatorname{cosec}(-x) &= -\operatorname{cosec} x & \cot(-x) &= -\cot x
\end{aligned}$$

$$\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x \qquad \operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\cot\left(\frac{\pi}{2} - x\right) = \tan x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x$$

$\frac{8\pi}{4}$

$\sec\left(\frac{11\pi}{4}\right)$

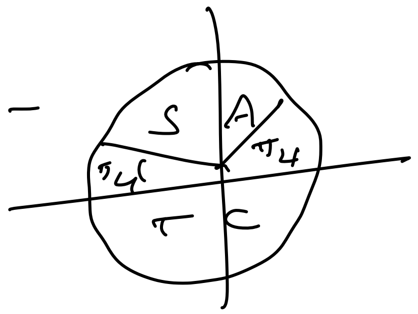
$= \sec\left(\cancel{\frac{8\pi}{4}} + \frac{3\pi}{4}\right)$

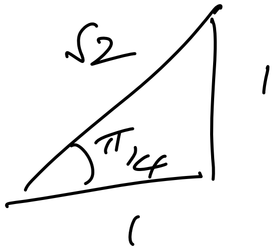
$= \sec\left(\frac{3\pi}{4}\right)$

$= \frac{1}{\cos\left(\frac{3\pi}{4}\right)}$

$= \frac{1}{\cos\left(-\frac{\pi}{4}\right)}$

$= \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$





Examples using the properties

Find the exact value of each of the following:

$\sec\left(\frac{11\pi}{4}\right)$

$\boldsymbol{cosec\left(-\frac{23\pi}{4}\right)}$

$\cot\left(\frac{11\pi}{3}\right)$

$$\begin{aligned}
\sec(\pi - x) &= -\sec x \\
\sec(\pi + x) &= -\sec x \\
\sec(2\pi - x) &= \sec x \\
\sec(-x) &= \sec x
\end{aligned}$$

$$\begin{aligned}
\operatorname{cosec}(\pi - x) &= \operatorname{cosec} x \\
\operatorname{cosec}(\pi + x) &= -\operatorname{cosec} x \\
\operatorname{cosec}(2\pi - x) &= -\operatorname{cosec} x \\
\operatorname{cosec}(-x) &= -\operatorname{cosec} x
\end{aligned}$$

$$\begin{aligned}
\cot(\pi - x) &= -\cot x \\
\cot(\pi + x) &= \cot x \\
\cot(2\pi - x) &= -\cot x \\
\cot(-x) &= -\cot x
\end{aligned}$$

$$\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x \qquad \operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$$

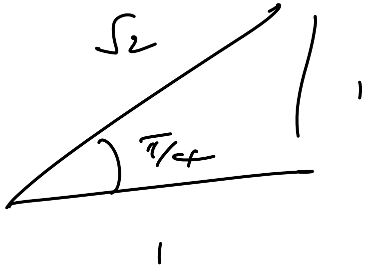
$$\cot\left(\frac{\pi}{2} - x\right) = \tan x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x$$

$\frac{8\pi}{4}$

$$\operatorname{cosec}\left(\frac{-23\pi}{4}\right)$$

$$= \operatorname{cosec}\left(\cancel{-\frac{24\pi}{4}} + \frac{\pi}{4}\right)$$

$$= \operatorname{cosec}\left(\frac{\pi}{4}\right) = \frac{1}{\sin\left(\frac{\pi}{4}\right)} = \frac{1}{\frac{1}{\sqrt{2}}} = \underline{\underline{\sqrt{2}}}$$



Examples using the properties

Find the exact value of each of the following:

$\sec\left(\frac{11\pi}{4}\right)$

$cosec\left(-\frac{23\pi}{4}\right)$

$\mathbf{cot}\left(\frac{\mathbf{11\pi}}{\mathbf{3}}\right)$

$$\sec(\pi - x) = -\sec x \qquad \text{cosec}\,(\pi - x) = \text{cosec}\ x \qquad \cot(\pi - x) = -\cot x$$

$$\sec(\pi + x) = -\sec x \qquad \text{cosec}\,(\pi + x) = -\text{cosec}\ x \qquad \cot(\pi + x) = \cot x$$

$$\sec(2\pi - x) = \sec x \qquad \text{cosec}\,(2\pi - x) = -\text{cosec}\ x \qquad \cot(2\pi - x) = -\cot x$$

$$\sec(-x) = \sec x \qquad \text{cosec}\,(-x) = -\text{cosec}\ x \qquad \cot(-x) = -\cot x$$

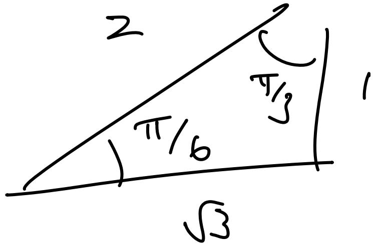
$$\sec\left(\frac{\pi}{2} - x\right) = \text{cosec}\ x \qquad \text{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\cot\left(\frac{\pi}{2} - x\right) = \tan x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x$$

$\frac{3\pi}{3}$

$$\cot\left(\frac{11\pi}{3}\right) = \cot\left(\cancel{\frac{12\pi}{3}} - \frac{\pi}{3}\right)$$

$$= \cot\left(-\frac{\pi}{3}\right) = \frac{1}{\tan\left(-\frac{\pi}{3}\right)} = \frac{1}{-\sqrt{3}} = \underline{\underline{-\frac{1}{\sqrt{3}}}}$$



A new identity – so much fun

We have already made extensive use of $\sin^2\theta + \cos^2\theta = 1$ in other areas of this course.

Using manipulation we could use it to get to

$\sec^2\theta - \tan^2\theta = 1$ by dividing everything by $\cos^2\theta$

We can now divide by $\sin^2\theta$ to get to the following:

$1 + \cot^2\theta = cosec^2\theta$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}}^{\,1} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\frac{\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}}^{\,1} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \frac{1}{\tan^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = cosec^2\theta$$

Trigonometric Identities and simplification

Using the new identity we can use it to simplify trig identities and expressions.

For example we can prove the following is the same as $\sin^3 x$

$$\frac{\cos x - \cos^3 x}{\cot x}$$

$$= \frac{\cos x\,(1 - \cos^2 x)}{\cot x}$$

$$= \cos x \cdot \sin^2 x \cdot \frac{1}{\cot x}$$

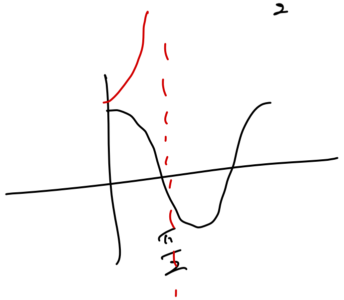
$$= \cancel{\cos x} \cdot \sin^2 x \cdot \frac{\sin x}{\cancel{\cos x}} = \underline{\underline{\sin^3 x}}$$

Examples

If $\tan x = 2$ and $x \in \left[0, \frac{\pi}{2}\right]$, find:

- $\sec x$
- $\cos x$
- $\sin x$
- $\operatorname{cosec} x$

$x \in [0, \frac{\pi}{2}]$



$$\sec^2 x = 1 + \tan^2 x$$
$$= 1 + 2^2$$
$$= 1 + 4$$
$$= 5$$

$$\sec x = \pm\sqrt{5}$$

$$\therefore \sec x = \sqrt{5}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cos x = \frac{1}{\sec x} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\therefore \sin x = \cos x \cdot \tan x = \frac{\sqrt{5}}{5} \cdot 2 = \frac{2\sqrt{5}}{5}$$

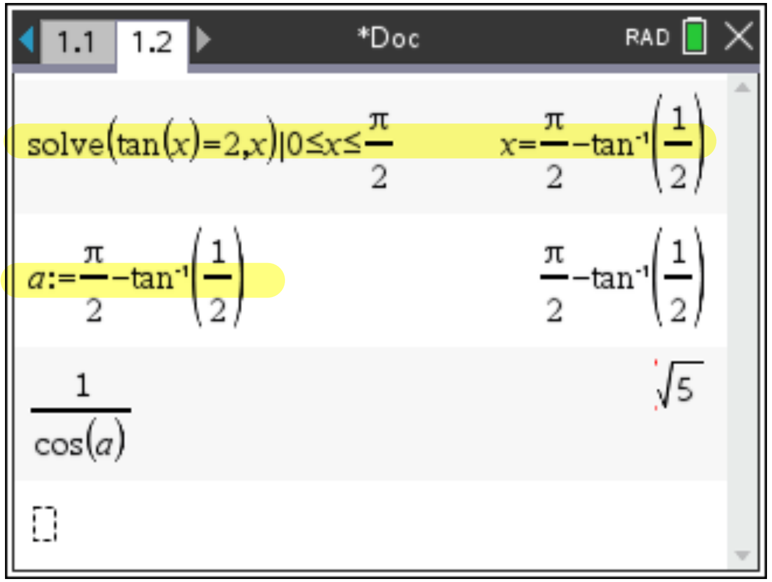
$$\operatorname{cosec} x = \frac{1}{\sin x}$$

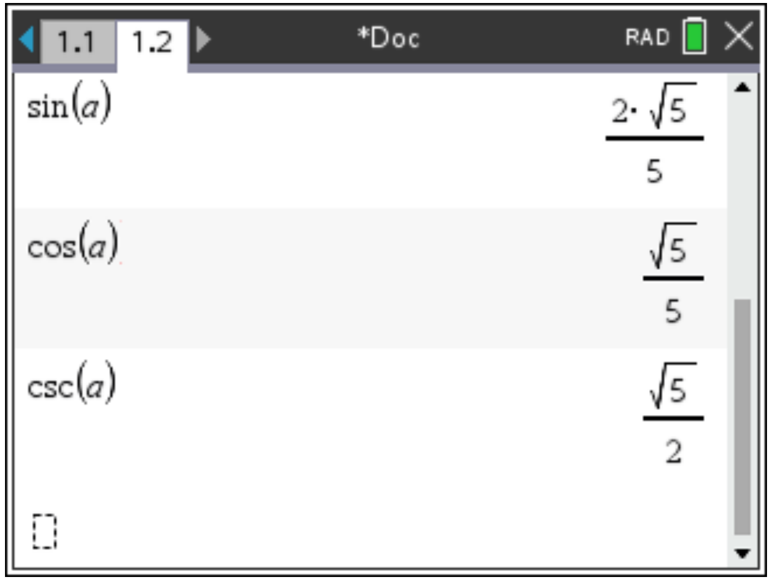
$$\operatorname{cosec} x = \frac{5}{2\sqrt{5}} = \frac{5\sqrt{5}}{10} = \frac{\sqrt{5}}{2}$$

Examples (using the CAS)

If $\tan x = 2$ and $x \in \left[0, \frac{\pi}{2}\right]$, find:

- $\sec x$
- $\cos x$
- $\sin x$
- $\operatorname{cosec} x$





Learning Objectives: Revisited

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what the cosecant function is and what it looks like
- Understand what the secant function is and what it looks like
- Understand what the cotangent function is and what it looks like
- Understand the useful properties of each of the above
- Understand how to use the CAS to solve functions containing the above.

Work to be completed

The following represents the minimum work which should be completed.

The more questions you answer from each exercise, chapter review and Checkpoints the better you chance of gaining an excellent study score in November.

Specialist Mathematics Units 3 and 4 Textbook

Chapter 3
Exercise 3A: The reciprocal circular functions
Questions: 1acf, 2ce, 3b, 4c, 5defijk, 6, 7, 9, 11, 13, 15, 16