

General Solution of Trigonometric Equations

Year 11
Mathematical Methods

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- Understand what it means to have a general solution for Trigonometric Equations
- Know how to find from first principles the general solutions.
- Be able to find the general solutions to equations
- Apply this understanding to a range of problems.

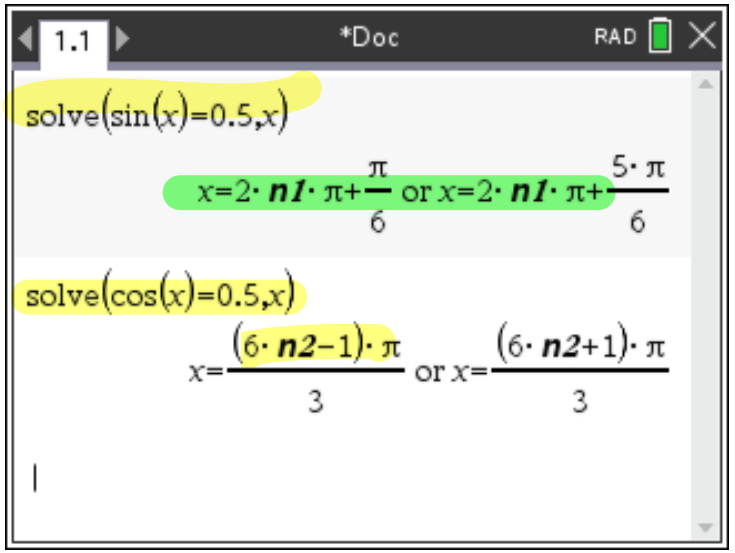


Recap of past learning

We have looked at a lot of content for Circular Functions. Being able to find solutions to trigonometric equations is really important. Generally, we limit the domain and hence you know how many solutions to find.

But what if we wanted a way to find all the solutions and express is in a nice algebraic way?

Have you already met this before? Perhaps with the CAS?



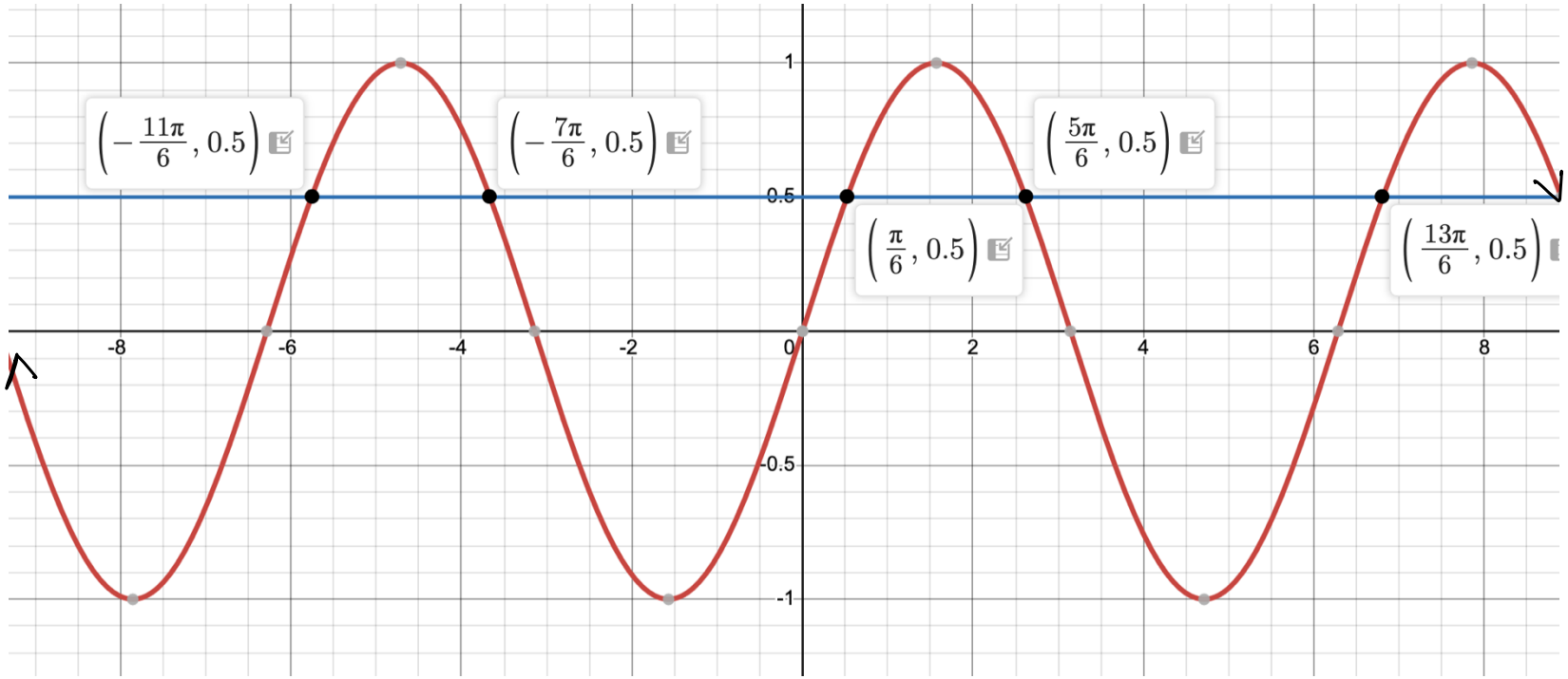
$\sin x = 0.5$



Finding the general solutions for COSINE

Let's look at finding the general solutions for $\cos x = 0.5$.

The first thing to notice is that there doesn't seem to be a domain! How many solutions am I looking to find then?



The actual solutions will extend to infinity and beyond!



Finding the general solutions for COSINE

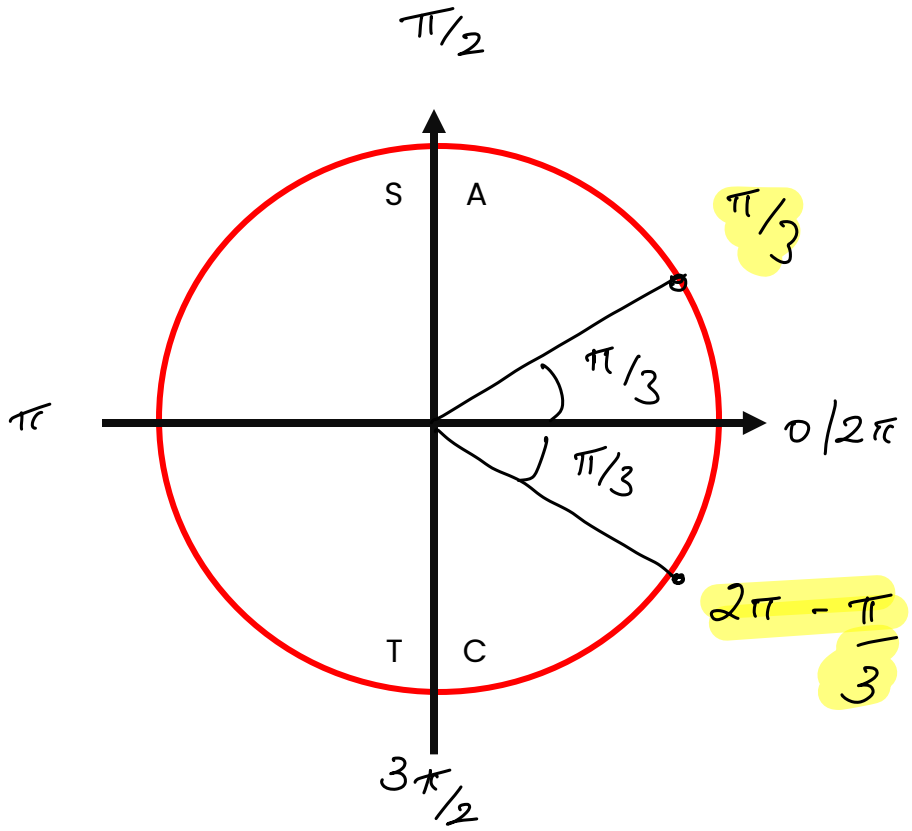
Let's look at finding the general solutions for $\cos x = 0.5$.

Step 1: Find one solution

The easiest way to do this would be to find:

$$\cos^{-1} 0.5 = \frac{\pi}{3}$$

And any other solution within the unit circle





Finding the general solutions for COSINE

Step 2: Add on 2π to each solution as, for COS, we know it repeats every 2π.

Rather than simplify the addition of the 2π leave it in the form shown. This will make a pattern much easier to spot.

Solution	Solution
$\frac{\pi}{3}$	$2\pi - \frac{\pi}{3}$
$2\pi + \frac{\pi}{3}$	$4\pi - \frac{\pi}{3}$
$4\pi + \frac{\pi}{3}$	$6\pi - \frac{\pi}{3}$
$6\pi + \frac{\pi}{3}$	$8\pi - \frac{\pi}{3}$



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$\frac{\pi}{3}$	
$2\pi + \frac{\pi}{3}$	$2\pi - \frac{\pi}{3}$
$4\pi + \frac{\pi}{3}$	$4\pi - \frac{\pi}{3}$
$6\pi + \frac{\pi}{3}$	$6\pi - \frac{\pi}{3}$
	$8\pi - \frac{\pi}{3}$



Finding the general solutions for COSINE

Step 2: Add on 2π to each solution as, for COS, we know it repeats every 2π.

Rather than simplify the addition of the 2π leave it in the form shown. This will make a pattern much easier to spot.

Solution	Solution
$0\pi + \frac{\pi}{3}$	$0\pi - \frac{\pi}{3}$
$2\pi + \frac{\pi}{3}$	$2\pi - \frac{\pi}{3}$
$4\pi + \frac{\pi}{3}$	$4\pi - \frac{\pi}{3}$
$6\pi + \frac{\pi}{3}$	$6\pi - \frac{\pi}{3}$
$8\pi + \frac{\pi}{3}$	$8\pi - \frac{\pi}{3}$



Finding the general solutions for COSINE

Now, let's align things in a different way and look at the coefficients of the start pi terms.

Solution	Solution
$0\pi + \frac{\pi}{3}$	$0\pi - \frac{\pi}{3}$
$2\pi + \frac{\pi}{3}$	$2\pi - \frac{\pi}{3}$
$4\pi + \frac{\pi}{3}$	$4\pi - \frac{\pi}{3}$
$6\pi + \frac{\pi}{3}$	$6\pi - \frac{\pi}{3}$
$8\pi + \frac{\pi}{3}$	$8\pi - \frac{\pi}{3}$



General equations to get odd an even numbers

We can use the expressions on the right to generate odd and even numbers.

This becomes really helpful.

$$2n, \qquad n \in \mathbb{Z}$$ ← **Even numbers**

$$2n + 1, \qquad n \in \mathbb{Z}$$ ← **Odd numbers**



Finding the general solutions for COSINE

We can replace the start numbers with the even number representation

$2n, \quad n \in \mathbb{Z}$

$2n+1, \quad n \in \mathbb{Z}$

Solution	Solution	
$0\pi + \frac{\pi}{3}$	$0\pi - \frac{\pi}{3}$	$2n\pi + \frac{\pi}{3}, \quad n \in \mathbb{Z}$
$2\pi + \frac{\pi}{3}$	$2\pi - \frac{\pi}{3}$	
$4\pi + \frac{\pi}{3}$	$4\pi - \frac{\pi}{3}$	$2n\pi - \frac{\pi}{3}, \quad n \in \mathbb{Z}$
$6\pi + \frac{\pi}{3}$	$6\pi - \frac{\pi}{3}$	
$8\pi + \frac{\pi}{3}$	$8\pi - \frac{\pi}{3}$	



Finding the general solutions for COSINE

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Which can be combined into one formula
(much like we do with the quadratic equation).

$$2n\pi + \frac{\pi}{3}, \quad n \in \mathbb{Z} \qquad\qquad 2n\pi - \frac{\pi}{3}, \quad n \in \mathbb{Z}$$

$$2n\pi \pm \frac{\pi}{3}, \quad n \in \mathbb{Z}$$

But this will only find the solutions for cos(x)=0.5



Finding the general solutions for COSINE

Which can be combined into one formula
(much like we do with the quadratic equation).

$\cos x = a$

$x = \cos^{-1}(a)$

$$2n\pi \pm \frac{\pi}{3}, \qquad n \in \mathbb{Z}$$

This came from the initial question. It is just one of the solutions. So, to find the solutions for any question we simply write it as:

$$x = 2n\pi \pm \cos^{-1} a, \qquad n \in \mathbb{Z}$$

This is the formula to find the general solutions for cosine functions



Finding the general solutions for COSINE

We can write the general solution for the equation $\cos x = 0.5$ in a better way

$$x = 2n\pi \pm \frac{\pi}{3}, \qquad n \in \mathbb{Z}$$

$$x = \frac{6n\pi}{3} \pm \frac{\pi}{3}, \qquad n \in \mathbb{Z}$$

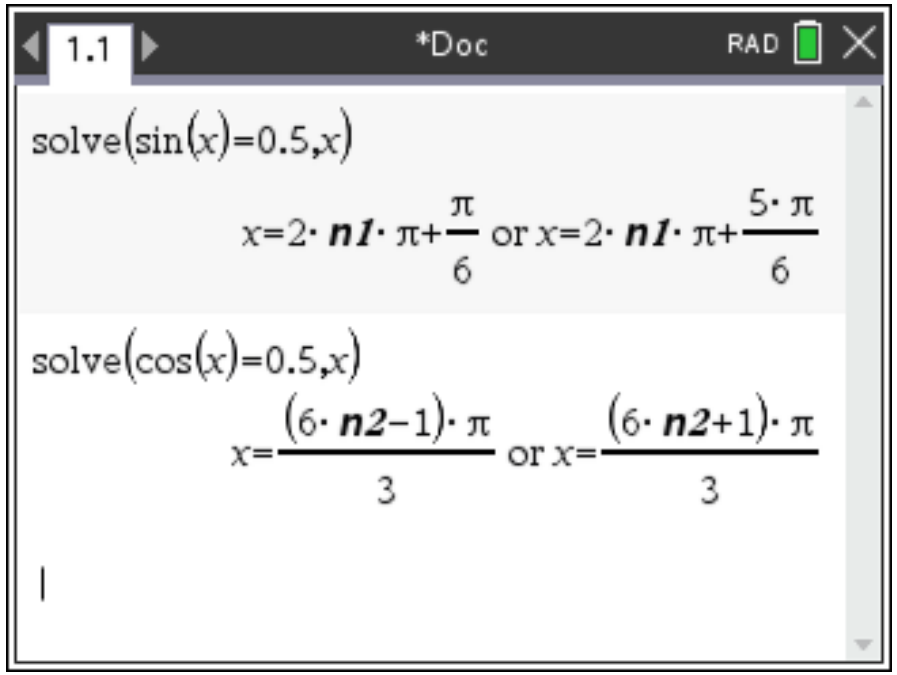
$$x = \frac{6n\pi \pm \pi}{3}, \qquad n \in \mathbb{Z}$$

$$x = \frac{\pi(6n \pm 1)}{3}, \qquad n \in \mathbb{Z}$$



Finding the general solutions for COSINE

We can write the general solution for the equation $\cos x = 0.5$ in a better way



$$x = \frac{\pi(6n+1)}{3}, \qquad n \in \mathbb{Z}$$

$$x = \frac{\pi(6n-1)}{3}, \qquad n \in \mathbb{Z}$$

Your CAS uses letters (n1, n2, n3, n4 …) to stand for parameters.
This is how you know you need to limit your domain if you don't want the general solution.



Finding the general solutions for TAN

We will do this a little quicker.

Remember that tan functions repeat every π.

Find the general solutions to:

$\tan x = 1$

First solution is $\frac{\pi}{4}$

$x = \tan^{-1}(1)$

$= \frac{\pi}{4}$

Solution	Solution
or $\frac{\pi}{4}$	$\pi + \frac{\pi}{4}$
$2\pi + \frac{\pi}{4}$	$3\pi + \frac{\pi}{4}$
$4\pi + \frac{\pi}{4}$	$5\pi + \frac{\pi}{4}$

The coefficients of pi are really just the counting numbers



Finding the general solutions for TAN

As these are the counting numbers we can now jump straight to the general solution.

Remember, this is the general solution to one specific problem.

$$x = n\pi + \frac{\pi}{4}, \qquad n \in \mathbb{Z}$$

Remember: The value of 'n' can be negative too.
If you are looking to find solutions which are negative.



Finding the general solutions for TAN

To make it a true general solution we simply write it as shown.

$\tan x = a$

$x = \tan^{-1}(a)$

$$x = n\pi + \tan^{-1} a\,, \qquad n \in \mathbb{Z}$$

This is simply one solution which satisfies the question given.



Finding the general solutions for SINE

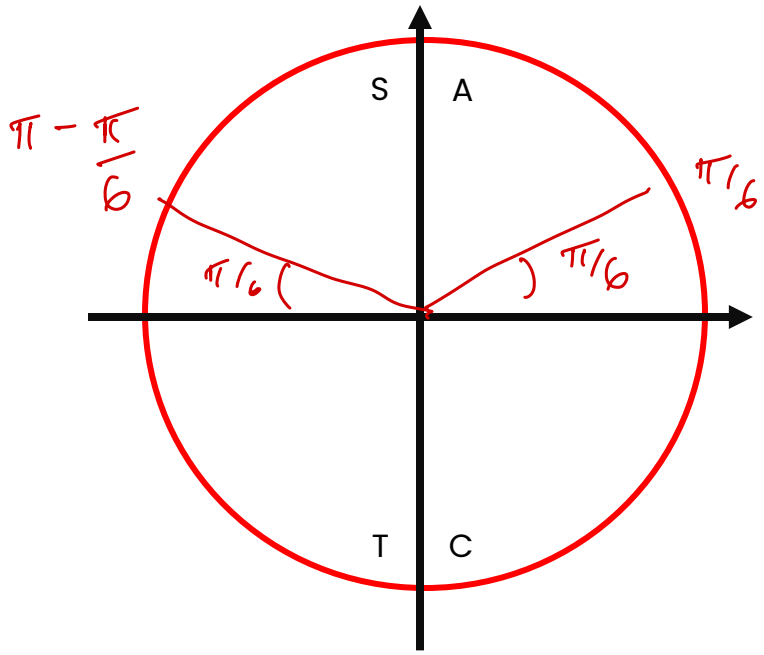
This is a little more complex but still easy to fo, once you understand the ideas I'm about to present.

Find the general solution to the following:

$$\sin x = 0.5$$

This has a simple first solution.

$$\sin^{-1} 0.5 = \frac{\pi}{6}$$



There is a second solution at:

$$\sin^{-1}(0.5) = \frac{5\pi}{6}$$



Finding the general solutions for SINE

This is a little more complex but still easy to find, once you understand the ideas I'm about to present.

Find the general solution to the following:

$$\sin x = 0.5$$

This has a simple first solution.

Solution	Solution
$\frac{\pi}{6}$	$\pi - \frac{\pi}{6}$
$2\pi + \frac{\pi}{6}$	$3\pi - \frac{\pi}{6}$
$4\pi + \frac{\pi}{6}$	$5\pi - \frac{\pi}{6}$
$6\pi + \frac{\pi}{6}$	$7\pi - \frac{\pi}{6}$



Finding the general solutions for SINE

If we look at the coefficients now … they seem to be the odd numbers on the right-hand side, and the even numbers on the left-hand side.

But we have met the formulae for this before.

Solution	Solution
$\pi + \frac{\pi}{6}$	$\pi - \frac{\pi}{6}$
$2\pi + \frac{\pi}{6}$	$3\pi - \frac{\pi}{6}$
$4\pi + \frac{\pi}{6}$	$5\pi - \frac{\pi}{6}$
$6\pi + \frac{\pi}{6}$	$7\pi - \frac{\pi}{6}$
$2n, \quad n \in \mathbb{Z}$	$2n + 1, \quad n \in \mathbb{Z}$
EVEN NUMBERS	**ODD NUMBERS**



Finding the general solutions for SINE

General Solution for this question only then becomes.

Sadly, there is no way to simplify this into one nice formula. So we have to remember two.

Again, in this instance, we would be able to write the solutions in a slightly nicer way.

Solution

$$2n\pi + \frac{\pi}{6}$$

Solution

$$(2n+1)\pi - \frac{\pi}{6}$$

Remember: 'n' can be negative!



Finding the general solutions for SINE

Making this two true General Solutions, we would have the formulae shown on the right.

$$x = 2n\pi + \sin^{-1} a, \qquad n \in \mathbb{Z}$$

$$x = (2n+1)\pi - \frac{\pi}{6}, \qquad n \in \mathbb{Z}$$

Remember: 'n' can be negative!

$x = (2n+1)\pi - \sin^{-1}(a)$



Example 1

Find the general solution of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

$\frac{\pi}{3}$

$\therefore x = 2n\pi \pm \frac{\pi}{3}$

$= \frac{6n\pi \pm \pi}{3}$

$= \frac{6n\pi \pm \pi}{3} = \frac{\pi(6n \pm 1)}{3}$

$x = 2n\pi + \sin^{-1} a, \quad n \in \mathbb{Z}$

$x = (2n+1)\pi - \sin^{-1} a, \quad n \in \mathbb{Z}$

$x = n\pi + \tan^{-1} a, \quad n \in \mathbb{Z}$

$x = 2n\pi \pm \cos^{-1} a, \quad n \in \mathbb{Z}$



Example 1

Find the general solution of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

$x = 2n\pi + \sin^{-1} a, \quad n \in \mathbb{Z}$

$x = (2n+1)\pi - \sin^{-1} a, \quad n \in \mathbb{Z}$

$x = n\pi + \tan^{-1} a, \quad n \in \mathbb{Z}$

$x = 2n\pi \pm \cos^{-1} a, \quad n \in \mathbb{Z}$

$\frac{\pi}{6}$

$\tan 3x = \frac{1}{\sqrt{3}}$

$3x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$3x = n\pi + \frac{\pi}{6}$

$x = \frac{n\pi}{3} + \frac{\pi}{18} = \frac{6n\pi}{18} + \frac{\pi}{18} = \frac{6n\pi + \pi}{18}$

$= \frac{\pi(6n+1)}{18}$



Example 1

Find the general solution of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

$\frac{\pi}{4}$

$2\sin x = \sqrt{2}$

$\sin x = \frac{\sqrt{2}}{2}$

$x = 2n\pi + \frac{\pi}{4}$

$x = (2n+1)\pi - \frac{\pi}{4}$

$x = 2n\pi + \sin^{-1} a, \quad n \in \mathbb{Z}$

$x = (2n+1)\pi - \sin^{-1} a, \quad n \in \mathbb{Z}$

$x = n\pi + \tan^{-1} a, \quad n \in \mathbb{Z}$

$x = 2n\pi \pm \cos^{-1} a, \quad n \in \mathbb{Z}$



Example 2

Find the first three positive solutions of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

$x = \dfrac{(6n \pm 1)\pi}{3}, \quad n \in \mathbb{Z}$

$n = 0 \quad x = \dfrac{\pi}{3} \quad -\dfrac{\pi}{3}$

$n = 1 \quad x = \dfrac{7\pi}{3} \quad \dfrac{5\pi}{3}$

$\therefore x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}$

$x = 2n\pi + \sin^{-1} a, \quad n \in \mathbb{Z}$

$x = (2n+1)\pi - \sin^{-1} a, \quad n \in \mathbb{Z}$

$x = n\pi + \tan^{-1} a, \quad n \in \mathbb{Z}$

$x = 2n\pi \pm \cos^{-1} a, \quad n \in \mathbb{Z}$



Example 2

Find the first three positive solutions of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

$x = 2n\pi + \sin^{-1} a, \quad n \in \mathbb{Z}$

$x = (2n+1)\pi - \sin^{-1} a, \quad n \in \mathbb{Z}$

$x = n\pi + \tan^{-1} a, \quad n \in \mathbb{Z}$

$x = 2n\pi \pm \cos^{-1} a, \quad n \in \mathbb{Z}$

$x = \dfrac{(6n+1)\pi}{18}, \quad n \in \mathbb{Z}$

$n = 0 \quad x = \dfrac{\pi}{18}$

$n = 1 \quad x = \dfrac{7\pi}{18}$

$n = 2 \quad x = \dfrac{13\pi}{18}$

$\therefore x = \dfrac{\pi}{18}, \dfrac{7\pi}{18}, \dfrac{13\pi}{18}$



Example 2

Find the first three positive solutions of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

$x = \dfrac{(8n+1)\pi}{4}, \quad n \in \mathbb{Z}$

$x = \dfrac{(8n+3)\pi}{4}, \quad n \in \mathbb{Z}$

$n = 0$ $\quad x = \dfrac{\pi}{4}$ $\quad x = \dfrac{3\pi}{4}$

$n = 1$ $\quad \dfrac{9\pi}{4}$ $\quad x = \dfrac{11\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}$

$x = 2n\pi + \sin^{-1} a, \quad n \in \mathbb{Z}$

$x = (2n+1)\pi - \sin^{-1} a, \quad n \in \mathbb{Z}$

$x = n\pi + \tan^{-1} a, \quad n \in \mathbb{Z}$

$x = 2n\pi \pm \cos^{-1} a, \quad n \in \mathbb{Z}$



Learning Objectives: Revisited

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- Understand what it means to have a general solution for Trigonometric Equations
- Know how to find from first principles the general solutions.
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