

Matching and allocation problems

Year 12 General Maths
Units 3 and 4

www.maffsguru.com

Recap of past learning

Just when you think you've learned all the language you need for this course, along comes lots more other words!

We've spent a lot of time looking at graphs and networks. We've found some practical applications of the theory.

Let's look at some more.



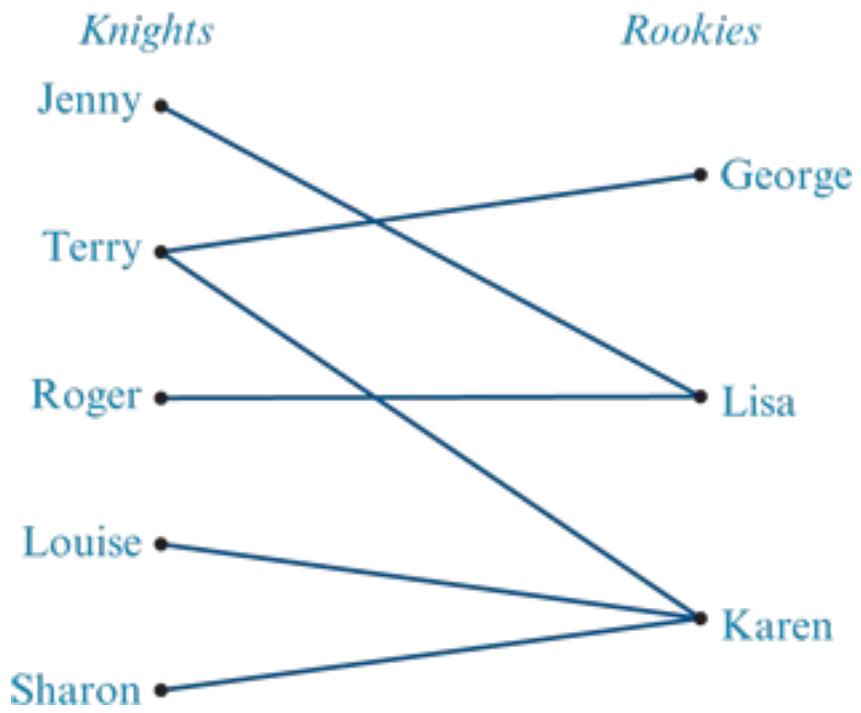
Bipartite graphs

I'm not a great lover of sports, but I do see the wonderful application of graphs and the theory to this area of life.

The diagram could represent different players on two chess teams. The lines which connect them show who needs to play whom.

This type of graph is called a **bipartite graph**.

Each edge in a bipartite graph joins one vertex from one group to a vertex in the other group.



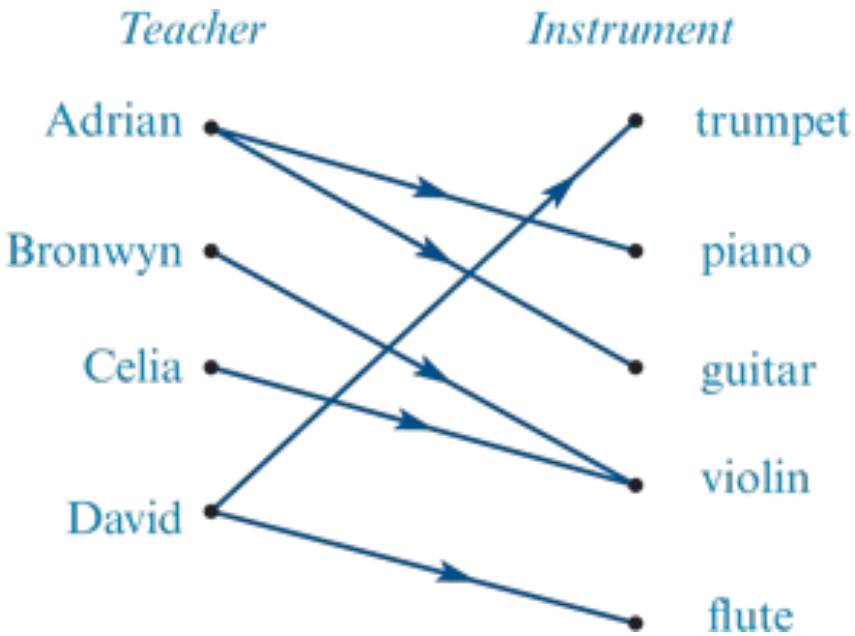


Directed bipartite graphs

I think it makes sense then that as soon as we add the word directed, there is going to be an arrow on the edge and this is going to stand for something important!

In matching problems, one vertex from one group is matched or allocated to one, or more, vertices of the second group and we use a directed bipartite graph to represent this matching.

This example looks at teachers and students.

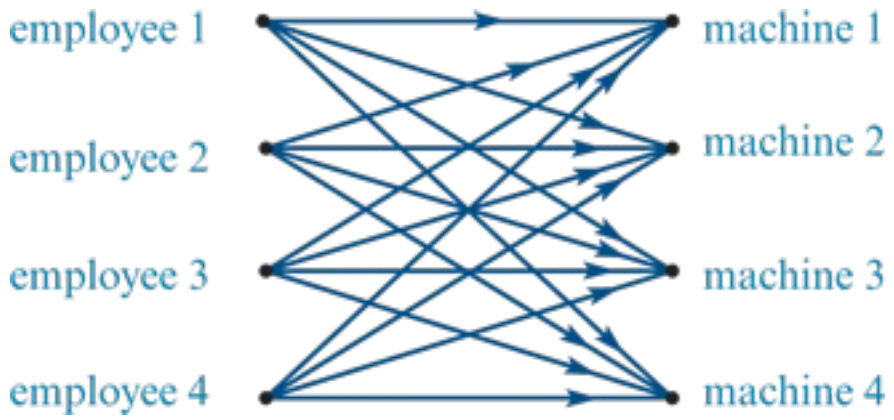




Complete weighted bipartite graphs

This is an example of a complete bipartite graph. It shows that each employee can work each machine in a factory. This would be good, but we know that people have strengths in different areas. It might well be that certain staff members are “quicker” at using one machine than others. To allocated the staff member to a machine they don’t really know how to use would lead to productivity loses.

It would be much better if we can find a way to match the correct employee to the correct machine to give us the most productive outcome.





The Hungarian Algorithm

The Hungarian Algorithm is used to allocate tasks to people.

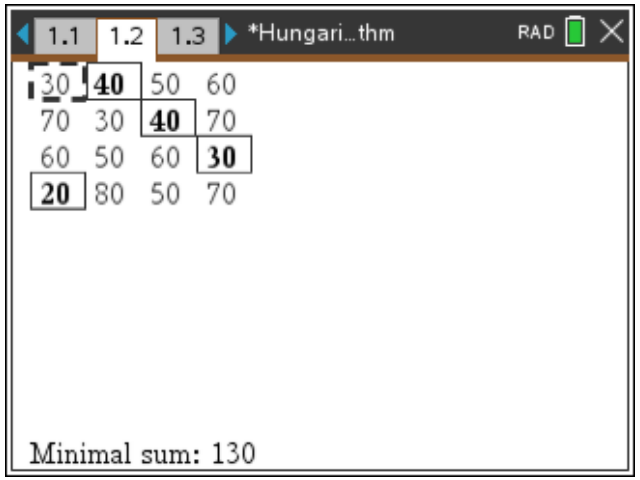
In the example shown, we have four tasks and four people. This is important.

To allocate using the Hungarian Algorithm you must have an equal number of tasks and people.

The Hungarian Algorithm is generally used to allocate tasks to people in the **MINIMUM** amount of time (or cost). You can use it to find the **MAXIMUM** but this is rarely used.

There is a CAS program which can be used to find the allocations. However, you are expected to be able to complete this by hand (showing all lines of working) in SACs.

Employee	*A*	*B*	*C*	*D*
Wendy	30	40	50	60
Xenefon	70	30	40	70
Yolanda	60	50	60	30
Zelda	20	80	50	70



Note: There might be more than one way to allocate. The CAS program only shows one.



The Hungarian Algorithm

The algorithm has a number of stages:

- Step 1: Subtract the lowest value in each row, from every value in each row
- Step 2: If the minimum number of lines required to cover all the zeros in the table is equal to the number of allocations to be made, jump to step 6. Otherwise, continue to step 3.
- Step 3: If a column does not contain a zero, subtract the lowest value in that column from every value in that column.
- Step 4: If the minimum number of lines required to cover all the zeros in the table is equal to the number of allocations to be made, jump to step 6. Otherwise, continue to step 5a.
- Step 5a: Add the smallest uncovered value to any value that is covered by two lines. Subtract the smallest uncovered value from all the uncovered values.
- Step 5b: Repeat from step 4.
- Step 6: Draw a directed bipartite graph with an edge for every zero value in the table.
- Step 7: Make the allocation and calculate minimum cost



Example

- Step 1: Subtract the lowest value in each row, from every value in each row
- Step 2: If the minimum number of lines required to cover all the zeros in the table is equal to the number of allocations to be made, jump to step 6. Otherwise, continue to step 3.
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The table below shows the four employees: Wendy, Xenefon, Yolanda and Zelda. The machines in a factory are represented by the letters A, B, C and D.
The numbers in the table are the times, in minutes, it takes each employee to finish the task on each machine.
The table is called a cost matrix.

Employee	*A*	*B*	*C*	*D*
Wendy	30	40	50	60
Xenefon	70	30	40	70
Yolanda	60	50	60	30
Zelda	20	80	50	70



Example

- Step 1: Subtract the lowest value in each row, from every value in each row
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Employee	*A*	*B*	*C*	*D*
Wendy	30	40	50	60
Xenefon	70	30	40	70
Yolanda	60	50	60	30
Zelda	20	80	50	70

	A	B	C	D
W	0	10	20	30
X	40	0	10	40
Y	30	20	30	0
Z	0	60	30	50



Example

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- Step 7: Make the allocation and calculate minimum cost

Employee	*A*	*B*	*C*	*D*
Wendy	0	10	20	30
Xenefon	40	0	10	40
Yolanda	30	20	30	0
Zelda	0	60	30	50

	A	B	C	D
W	0	10	10	30
X	40	0	0	40
Y	30	20	20	0
Z	0	60	20	50



Example

- Step 1: Subtract the lowest value in each row, from every value in each row
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Wendy	0	10	10	30
Xenefon	40	0	0	40
Yolanda	30	20	20	0
Zelda	0	60	20	50

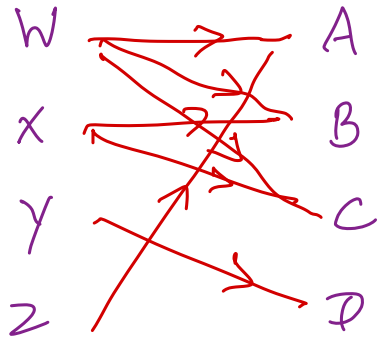
	A	B	C	D
W	0	0	0	30
X	50	0	0	50
Y	30	10	10	0
Z	0	50	10	50



Example

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Employee	*A*	*B*	*C*	*D*
Wendy	0	0	0	30
Xenefon	50	0	0	50
Yolanda	30	10	10	0
Zelda	0	50	10	50

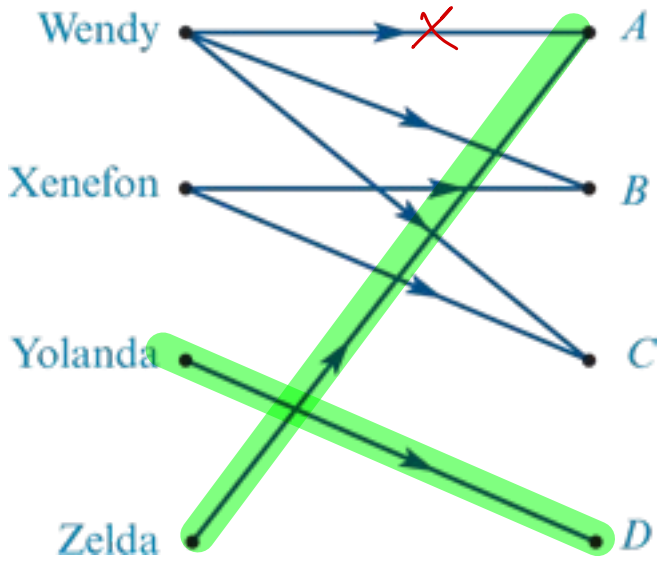




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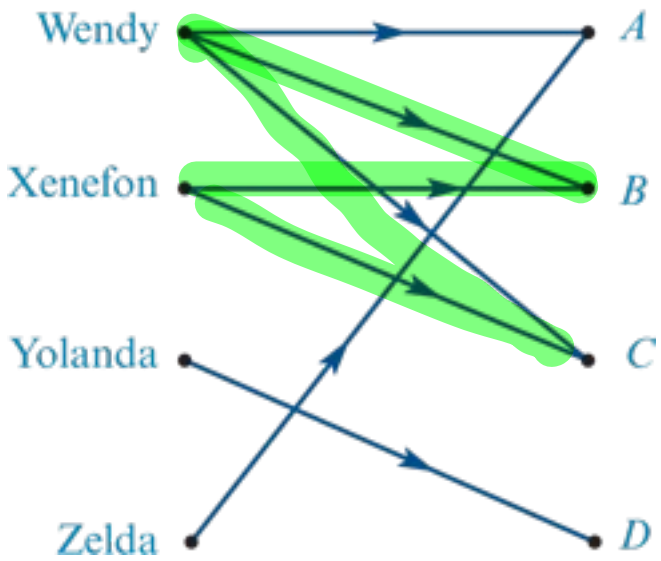
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Wendy	30	40	50	60
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Yolanda	60	50	60	30
Zelda	20	80	50	70





Example

Employee	*A*	*B*	*C*	*D*
Wendy	30	40	50	60
Xenefon	70	30	40	70
Yolanda	60	50	60	30
Zelda	20	80	50	70



W → B 40

X → C 40

Z → A 20

Y → D 30

130

W → C 50

X → B 30

Z → A 20

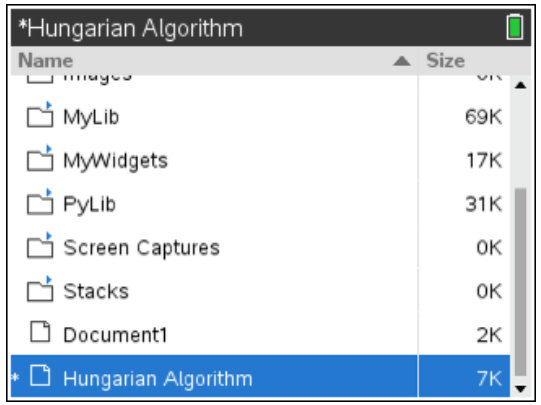
Y → D 30

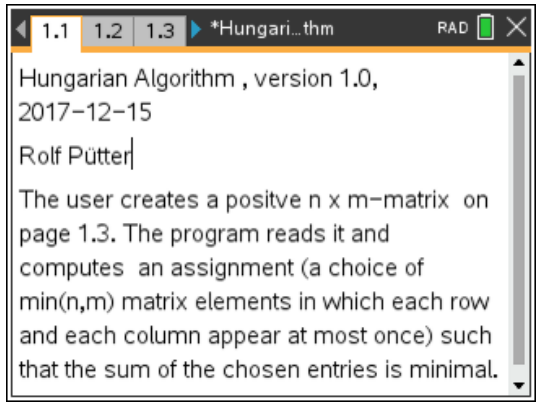
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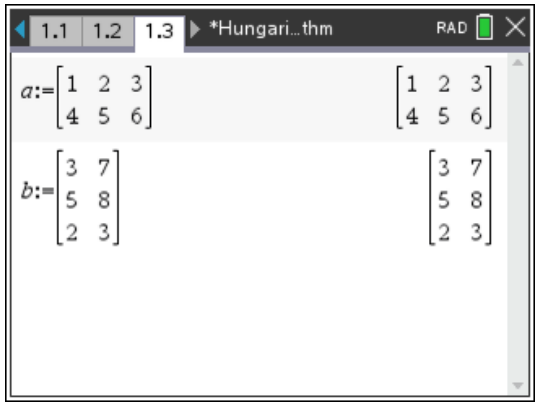


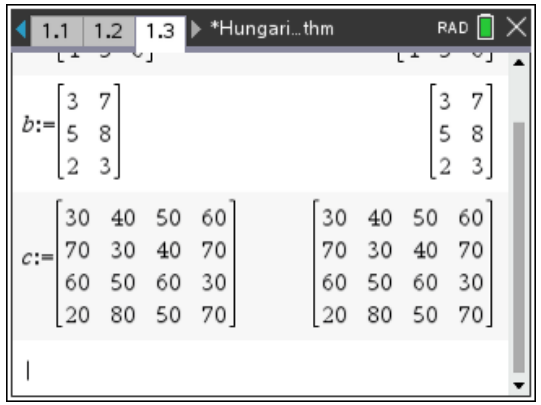
Using the CAS

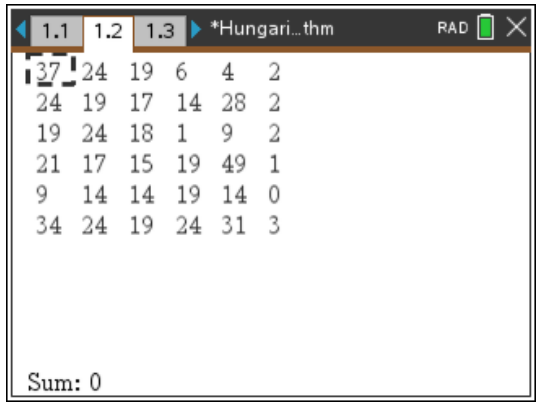
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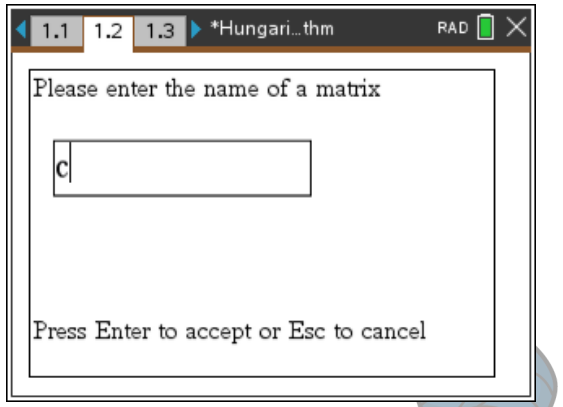








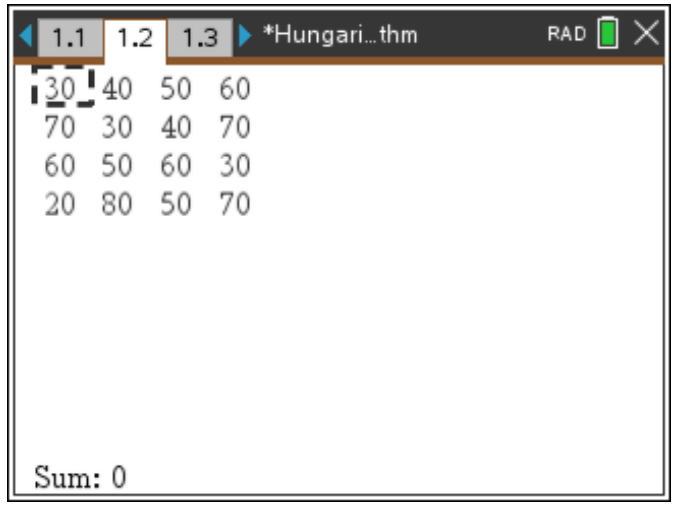


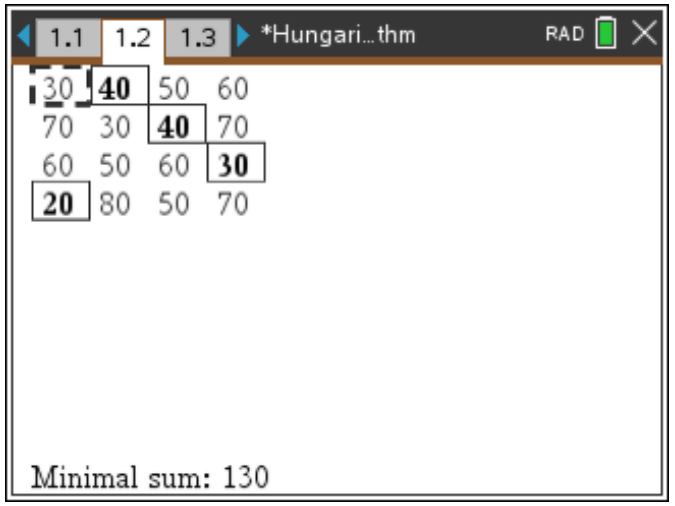




Using the CAS

Employee	*A*	*B*	*C*	*D*
Wendy	30	40	50	60
Xenefon	70	30	40	70
Yolanda	60	50	60	30
Zelda	20	80	50	70





Note: There are detailed instructions in the Summary Book showing you how to use this.



Finding the maximum

The way to find the maximum would be to follow these two steps:

- Find the maximum value in the matrix and subtract each element in the matrix from this maximum value.
- Now complete the algorithm with the minimisation procedure

Employee	*A*	*B*	*C*	*D*
Wendy	30	40	50	60
Xenefon	70	30	40	70
Yolanda	60	50	60	30
Zelda	20	80	50	70

80

	A	B	C	D
W	50	40	30	20
X	10	50	40	10
Y	20	30	20	50
Z	60	0	30	10



VCAA Questions

Question 7

Four friends go to an ice-cream shop.

Akiro chooses chocolate and strawberry ice cream.

Doris chooses chocolate and vanilla ice cream.

Gohar chooses vanilla ice cream.

Imani chooses vanilla and lemon ice cream.

This information could be presented as a graph.

Consider the following four statements:

- The graph would be connected.
- The graph would be bipartite.
- The graph would be planar.
- The graph would be a tree.

How many of these four statements are true?

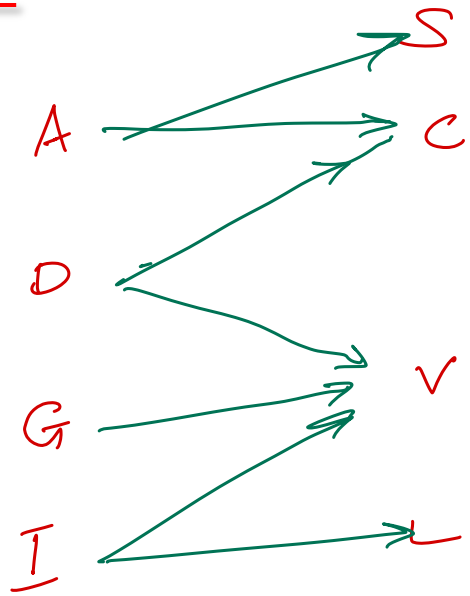
A. 0

B. 1

C. 2

D. 3

E. 4





VCAA Questions

Question 2 (1 mark)

A cricket team has 11 players who are each assigned to a batting position.

Three of the new players, Alex, Bo and Cameron, can bat in position 1, 2 or 3.

The table below shows the average scores, in runs, for each player for the batting positions 1, 2 and 3.

		Batting position		
		1	**2**	**3**
Player	**Alex**	22	24	24
	Bo	25	~~25~~	21
	Cameron	24	25	19

25

Each player will be assigned to one batting position.

To which position should each player be assigned to **maximise** the team's score? Write your answer in the table below.

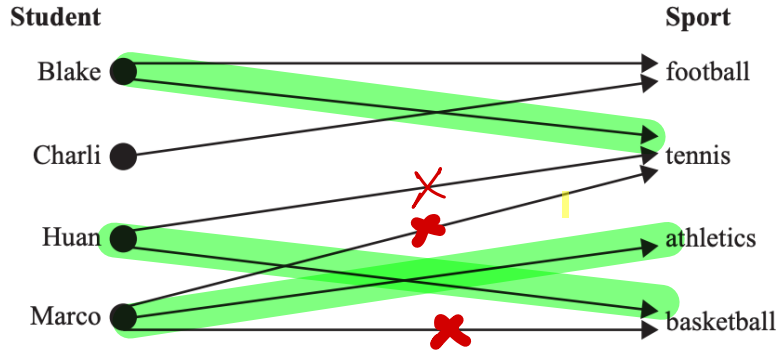
Player	Batting position
Alex	3
Bo	1
Cameron	2



VCAA Questions

Question 2 (3 marks)

Fencedale High School offers students a choice of four sports, football, tennis, athletics and basketball. The bipartite graph below illustrates the sports that each student can play.



Each student will be allocated to only one sport.

a. Complete the table below by allocating the appropriate sport to each student. 1 mark

Student	Sport
Blake	Tennis
Charli	Football
Huan	Basketball
Marco	Athletics



VCAA Questions

b. The school medley relay team consists of four students, Anita, Imani, Jordan and Lola.
The medley relay race is a combination of four different sprinting distances: 100 m, 200 m, 300 m and 400 m, run in that order.
The following table shows the best time, in seconds, for each student for each sprinting distance.

	Best time for each sprinting distance (seconds)			
Student	**100 m**	**200 m**	**300 m**	**400 m**
Anita	13.3	29.6	61.8	87.1
Imani	14.5	29.6	63.5	88.9
Jordan	13.3	29.3	63.6	89.1
Lola	15.2	29.2	61.6	87.9

The school will allocate each student to one sprinting distance in order to minimise the total time taken to complete the race.

To which distance should each student be allocated?
Write your answers in the table below. 2 marks

Student	**Sprinting distance (m)**
Anita	400
Imani	200
Jordan	100
Lola	300



VCAA Questions

Question 2 (2 marks)

Bai joins his friends Agatha, Colin and Diane when he arrives for the holiday in Seatown.
Each person will plan one tour that the group will take.
Table 1 shows the time, in minutes, it would take each person to plan each of the four tours.

Table 1

	Agatha	Bai	Colin	Diane
Tour 1	13	7	13	12
Tour 2	14	9	8	7
Tour 3	19	25	21	18
Tour 4	10	7	11	10

The aim is to minimise the total time it takes to plan the four tours.
Agatha applies the Hungarian algorithm to Table 1 to produce Table 2.
Table 2 shows the final result of all her steps of the Hungarian algorithm.

Table 2

	Agatha	Bai	Colin	Diane
Tour 1	3	0	3	3
Tour 2	6	4	0	0
Tour 3	0	9	2	0
Tour 4	0	0	1	1

a. In Table 2 there is a zero in the column for Colin.

When all values in the table are considered, what conclusion about minimum total planning time can be made from this zero? 1 mark

b. Determine the minimum total planning time, in minutes, for all four tours. 1 mark



VCAA Questions: 2024 Paper 1

Question 39

Anush, Blake, Carly and Dexter are workers on a construction site. They are each allocated one task.

The time, in hours, it takes for each worker to complete each task is shown in the table below.

	Task 1	Task 2	Task 3	Task 4
Anush	12	8	16	9
Blake	10	7	15	10
Carly	11	10	18	12
Dexter	10	14	16	11

The tasks must be completed sequentially and in numerical order: Task 1, Task 2, Task 3 and then Task 4.

Management makes an initial allocation of tasks to minimise the amount of time required, but then decides that it takes the workers too long.

Another worker, Edgar, is brought in to complete one of the tasks.

His completion times, in hours, are listed below.

	Task 1	Task 2	Task 3	Task 4
Edgar	9	5	14	8

When a new allocation is made and Edgar takes over one of the tasks, the minimum total completion time compared to the initial allocation will be reduced by

A. 1 hour.

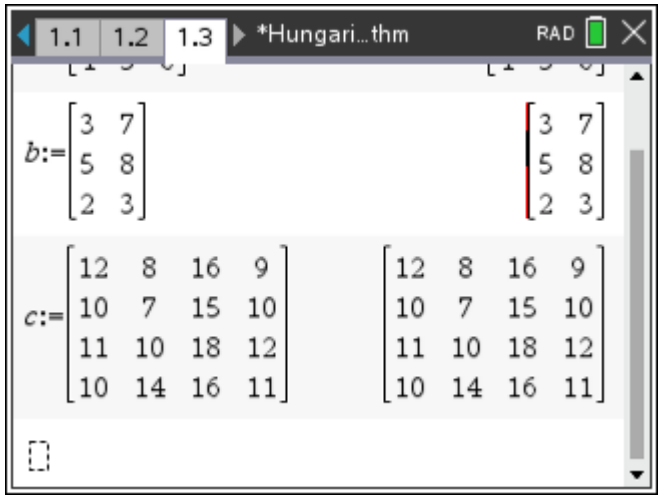
B. 2 hours.

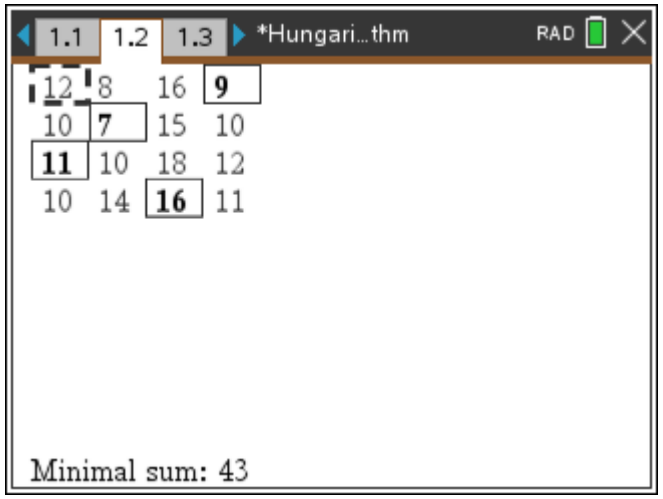
C. 3 hours.

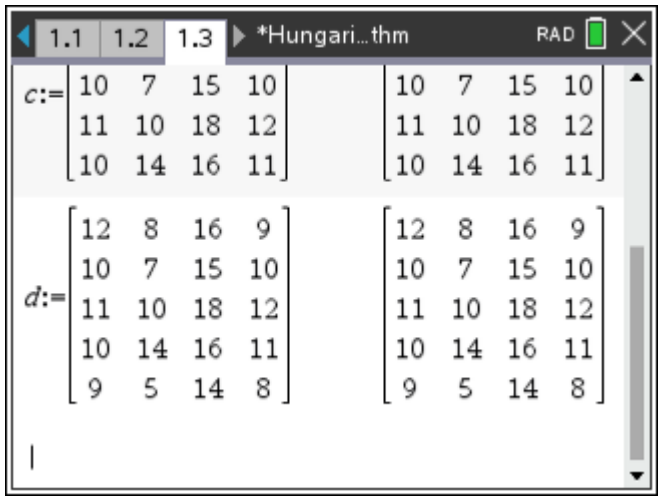
D. 4 hours.

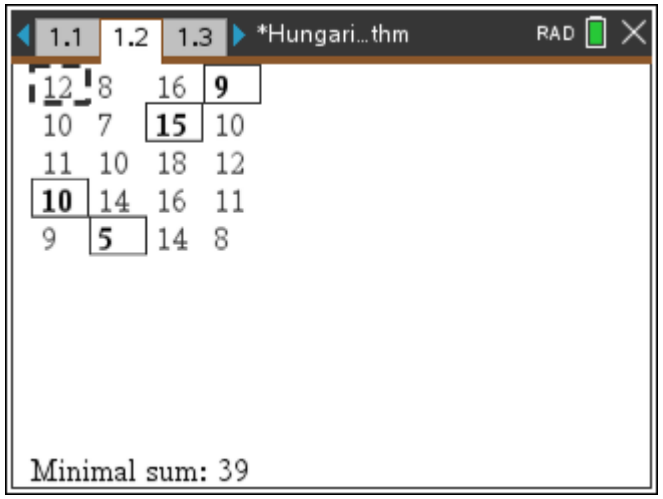


VCAA Questions: 2024 Paper 1











VCAA Questions: 2024 Paper 1 (By Hand)

Question 39

Anush, Blake, Carly and Dexter are workers on a construction site. They are each allocated one task.

The time, in hours, it takes for each worker to complete each task is shown in the table below.

	Task 1	Task 2	Task 3	Task 4
Anush	12	8	16	9
Blake	10	7	15	10
Carly	11	10	18	12
Dexter	10	14	16	11
Edgar	9	5	14	8

Task 5

0

0

0

0

0



VCAA Questions: 2024 NHT Paper 1 (Maximising)

Question 36

Four students, Peggy, Quincy, Radley and Sarah, are grouped together to complete a project. The project is in four parts, labelled W, X, Y and Z. Each student must complete one part of the project.

The table below shows each student's estimate of the score they will receive if they complete each section.

	Peggy	Quincy	Radley	Sarah
W	12	19	18	16
X	16	15	15	16
Y	10	16	17	15
Z	19	20	18	18

Based on the estimates, which allocation of project parts will maximise the students' group score on the project?

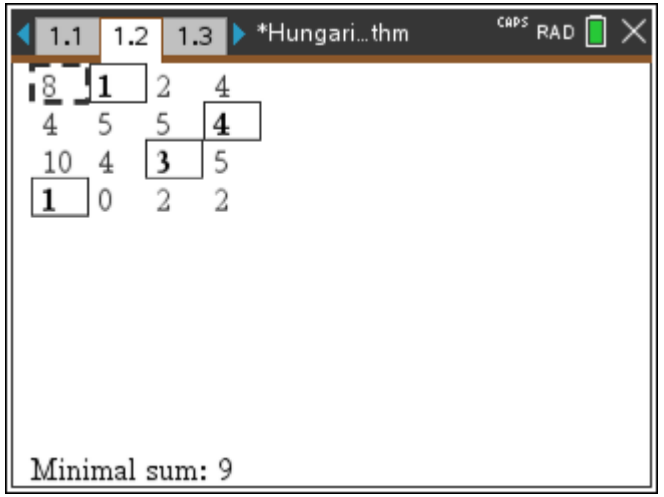
To maximise, they would need to do the tasks allocated as shown in this table:

A.

W	Quincy
X	Sarah
Y	Radley
Z	Peggy



Highest number is 20, so subtract all numbers in the table from 20. Then do algorithm as normal.



Questions to complete

All questions should be completed from the chapter related to this exercise

