Flow problems

Year 12 General Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Further Mathematics course.

- Understand what a directed graph is
- Understand what minimum flow means
- Know what a cut is and how to apply it to a directed graph
- Understand what it means by the capacity of a cut
- Understand what it means by minimum cut capacity

Recap of past learning

Putting aside the idea that flow problems could be something completely different in the medical world, this is building on the learning from the last lesson where we met weighted graphs. These were graphs where the edges had numbers assigned to them. These numbers could stand for the distance between two towns for example or the time taken to travel between towns etc.

No accounting has been made for the direction in which traffic might be moving for flowing!

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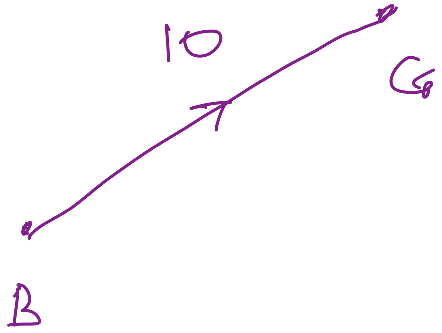
Directed graphs

We know, from experience, that roads can be one-way and two way. When we assign a direction to an edge we give it another name …

A graph represents connections between things.

It is important to note that **when a graph has numerical information we call it a network**.

When we then add a direction to the graph we call it a **directed graph** or **digraph** for short!



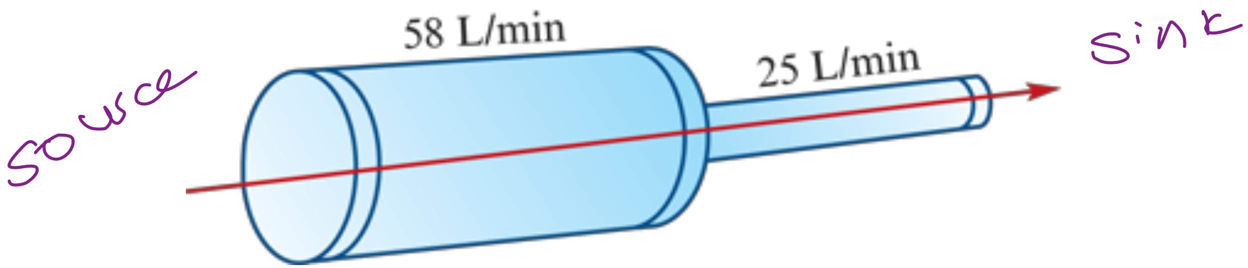
Minimum flow = smallest capacity

The best way I can describe this is to think about getting up in the middle of the night simple bursting to go to the bathroom. Your body has known for some time that it needs to go, but it was really enjoying the dream it was having. In the time you were dreaming your body did all it could to stop you from going to the toilet.

When it was finally time, you sit there for a time and wonder why it is taking so long to actually go! It should be flooding out of you like a jet stream.

Well, this has everything to do with flow!

If you think of the following … it doesn't matter that the bigger section of the pipe has 58L/min … the only thing which is getting through is going to be 25 L/min.

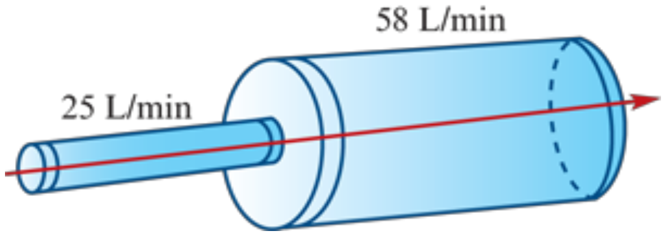


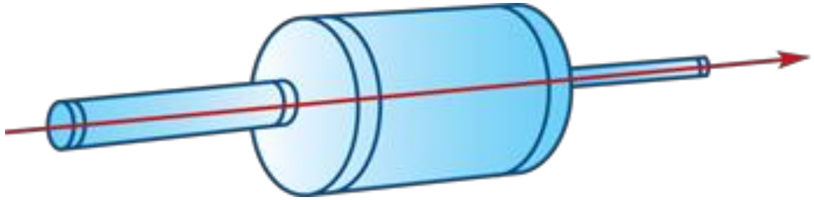
Minimum flow = smallest capacity

This is the whole concept of flow and minimum flow through a directed graph.

We must always think of flow as going from the **source** to the **sink**.

What will happen with the next examples?





Maximum flow

It doesn't matter how many pipes are connected together, the **maximum flow** through a series of pipes is going to be the minimum capacity of the individual pipes.

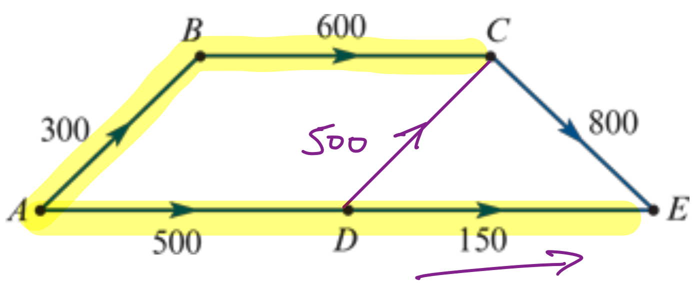
This is going to be really important!

max flow = min cut...

Example

n the digraph shown on the right, the vertices A, B, C, D and E represent towns. The edges of the graph represent roads and the weights of those edges are the maximum number of cars that can travel on the road each hour. The roads allow only one-way travel.

a) Find the maximum traffic flow from A to E through town C.
b) Find the maximum traffic flow from A to E overall.
c) A new road is being built to allow traffic from town D to town C. This road can carry 500 cars per hour.
   i. Add this road to the digraph.
   ii. Find the maximum traffic flow from A to E overall after this road is built.



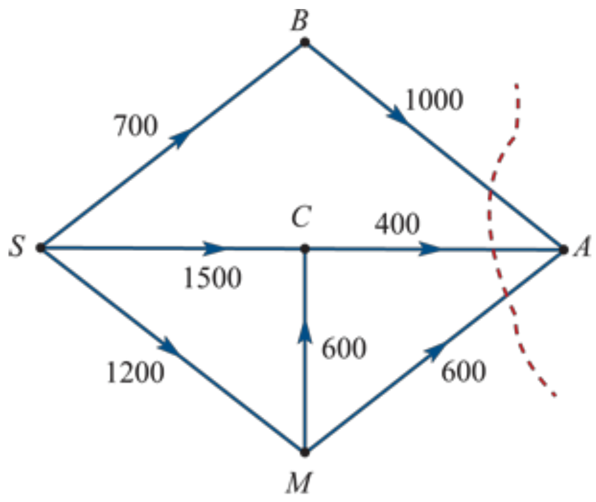
Cuts

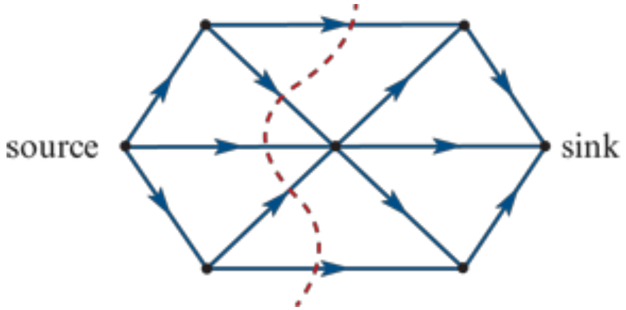
It isn't always easy, by inspection, to determine the maximum flow for a directed network.

If we have lots of vertices and edges, we need to look at something called a **cut**.

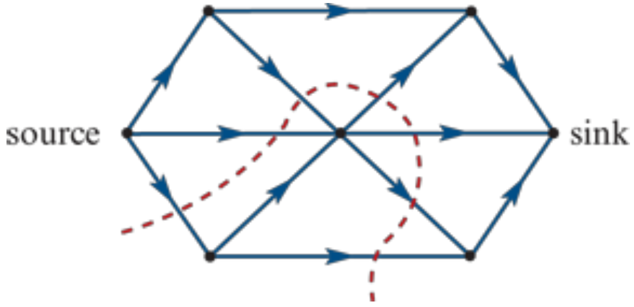
A cut basically divides a network into two parts. **It is really important to note that a cut must separate the source from the sink**!

A cut is shown with a dotted line.





A good cut

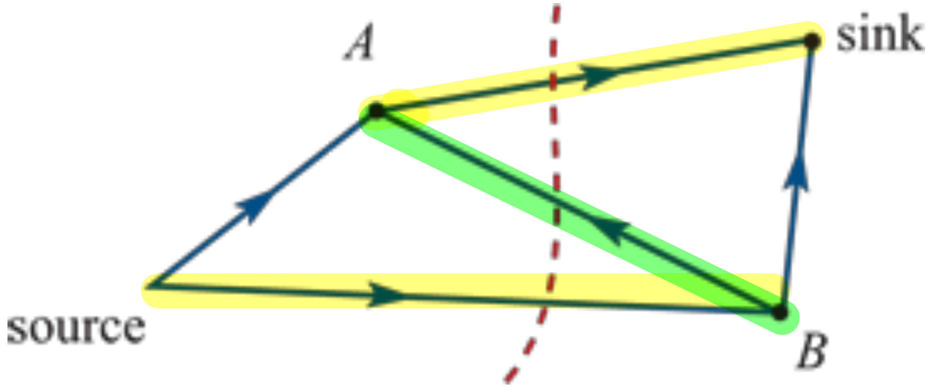


A bad cut

Capacity of a cut

We are really interested in knowing the capacity of a cut.

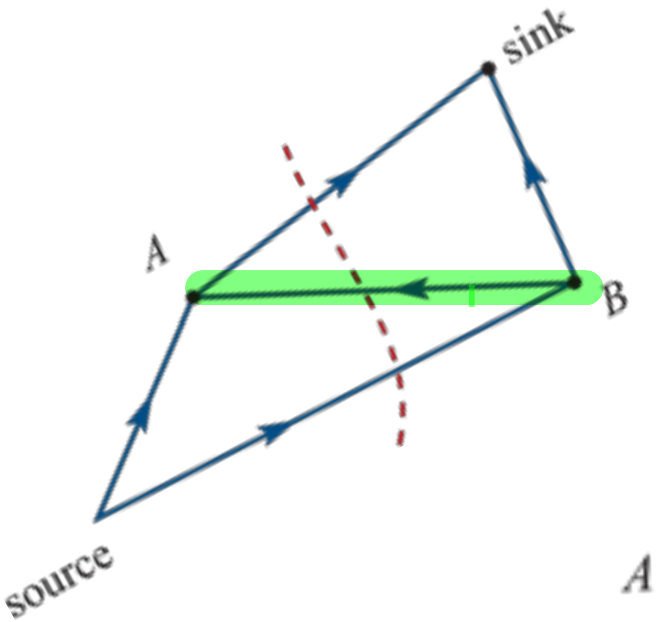
It's defined as the sum of all the capacities of the edges the cut passes through. A cut is only counted when the edge passes through it in the direction from the source to the sink.

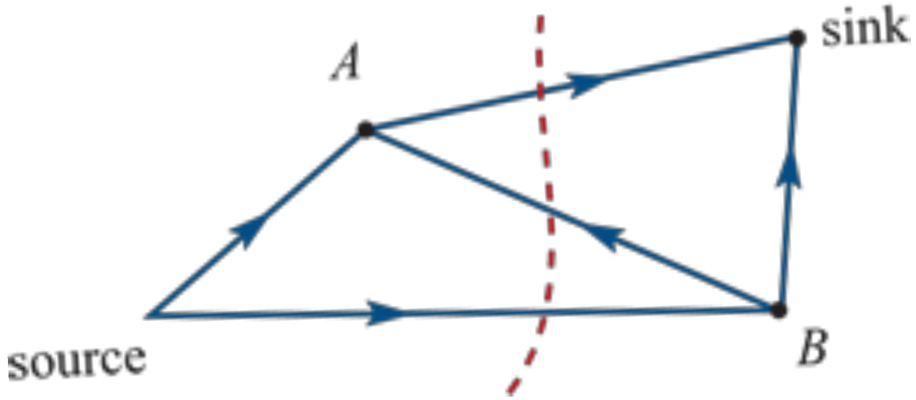


Can you spot the edge which wouldn't be counted?

Capacity of a cut – Turn the page!

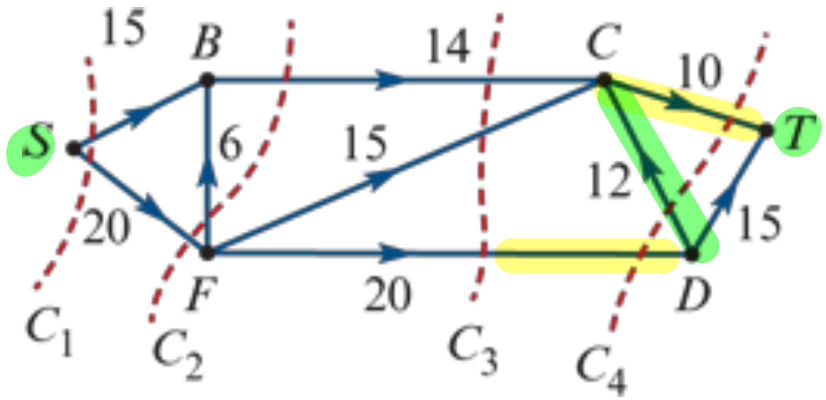
It can be much easier … when doing these questions on a physical media, to simply turn the page so your edge is horizontal. Then look and see if the arrow is points from right to left (which would be a bad cut) or left to right (which would be a good cut).





Example

Calculate the capacity of the four cuts shown in the network on the right. The source is vertex S and the sink is vertex T.



$C_1 = 35$ ✓

$C_2 = 14 + 20$

$= 34$ ✓

$C_3 = 14 + 15 + 20$

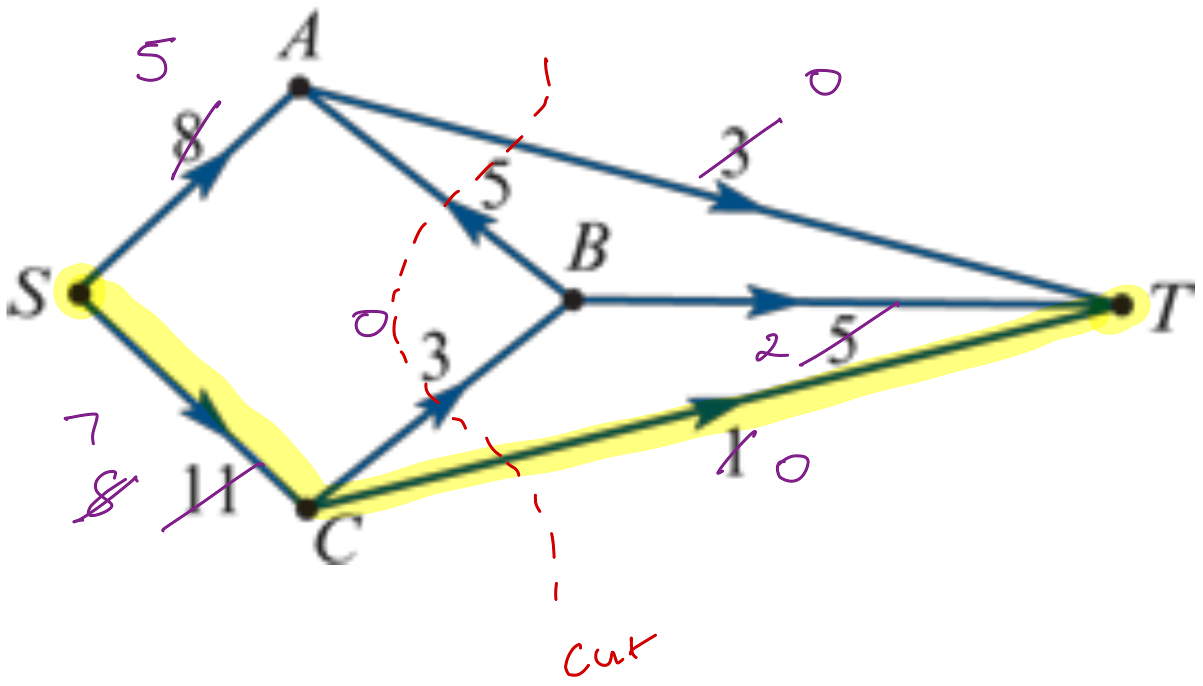
$= 49$ ✓

$C_4 = 30$

Example: Minimum cut capacity

Note: The minimum cut capacity possible for a graph equals the maximum flow through the graph.

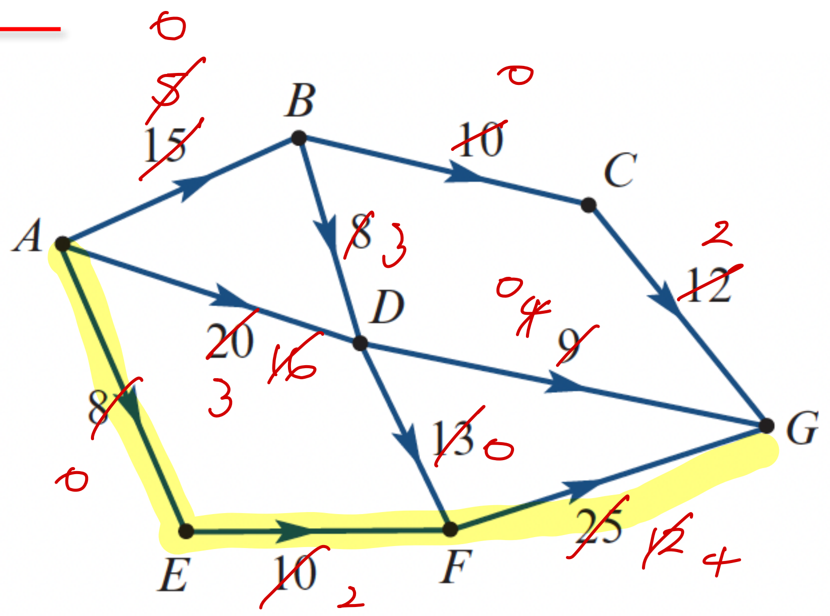
Example: Determine the maximum flow from S to T for the digraph shown on the right.



3
3
1
7

Example: Minimum cut capacity

The koala sanctuary in Cowes allows visitors to walk through their park. The park is represented by a network below, where each edge represents one-way tracks for visitors through the park. The direction of travel on each track is shown by an arrow. The numbers on the edges indicate the maximum number of people who are permitted to walk along each track each hour.



$10 + 5 + 4 + 13 + 8 = 40$

a Starting at A, how many people are permitted to walk to G each hour?

b Given that one group of nine people would like to walk from A to G together as a group, list all the different routes they could take so that the entire group of nine will stay together for the duration of their walk.

c What is the largest group of people that could walk through the koala sanctuary if they must stay together in a group for the entire duration of the walk?

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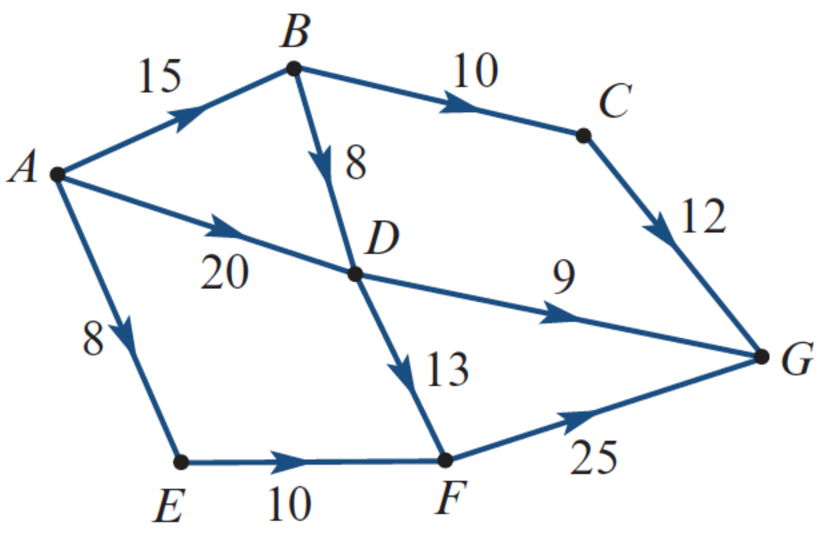

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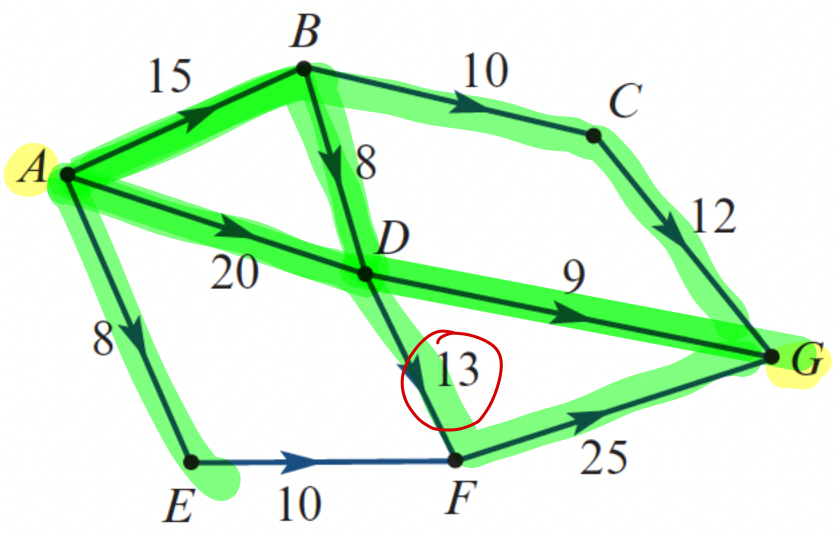

A B C G

A D G   A D F G

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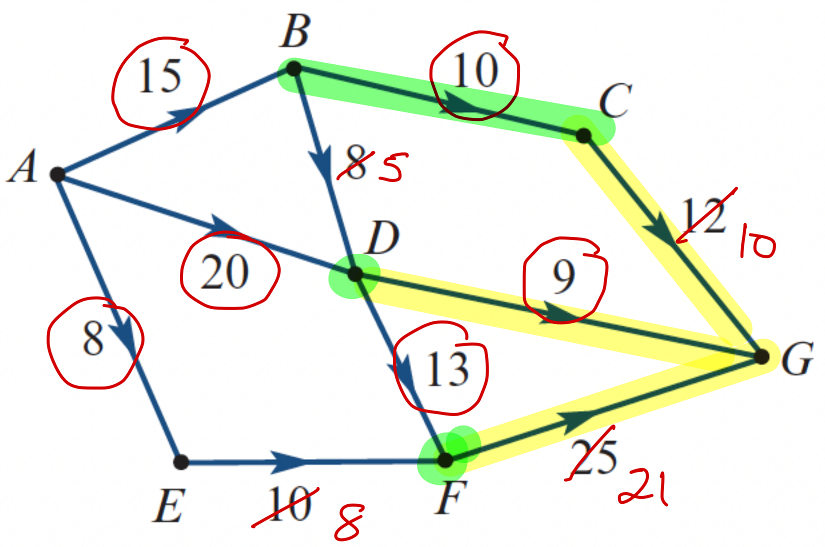

40

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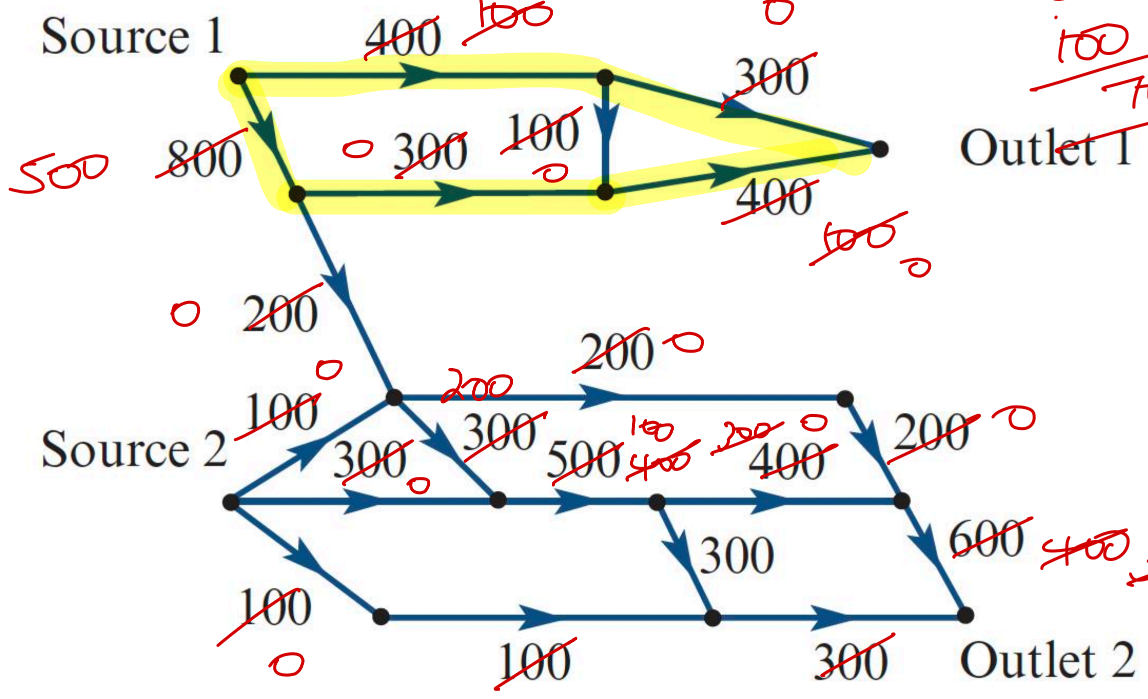
Example

Storm water enters a network of pipes at either Dunlop North (Source 1) or Dunlop South (Source 2) and flows into the ocean at either Outlet 1 or Outlet 2.

On the network diagram below, the pipes are represented by straight lines with arrows that indicate the direction of the flow of water. Water cannot flow through a pipe in the opposite direction.

The numbers next to the arrows represent the maximum rate, in kilolitres per minute, at which storm water can flow through each pipe.

Determine the maximum rate, in kilolitres per minute, at which water can flow from these pipes into the ocean at Outlet 1 and Outlet 2.

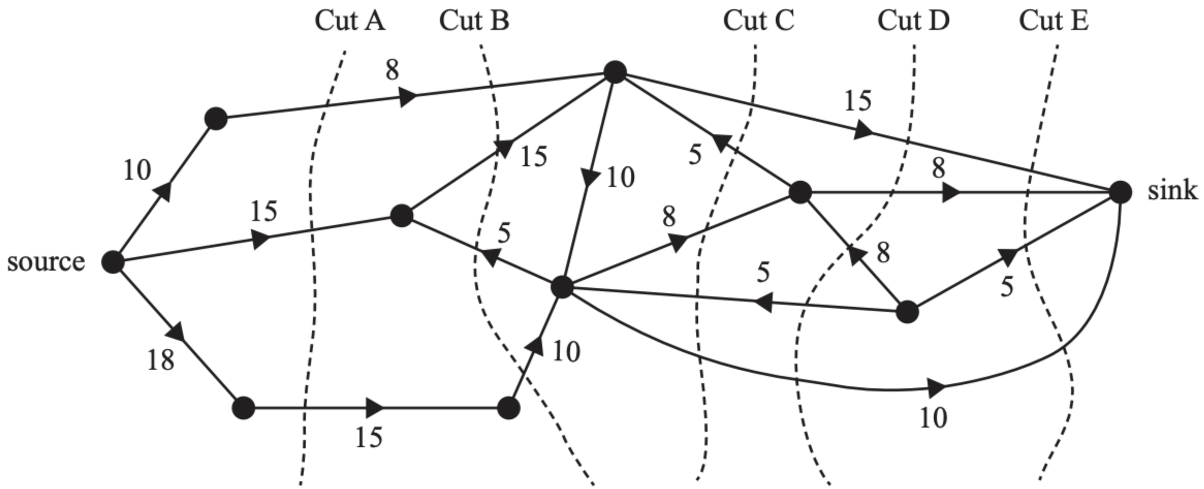


VCAA Question

This question has been taken from the VCAA 2020 FM Paper 1

Question 9

The flow of liquid through a series of pipelines, in litres per minute, is shown in the directed network below.



Five cuts labelled A to E are shown on the network.

The number of these cuts with a capacity equal to the maximum flow of liquid from the source to the sink, in litres per minute, is

A. 1

B. 2

C. 3

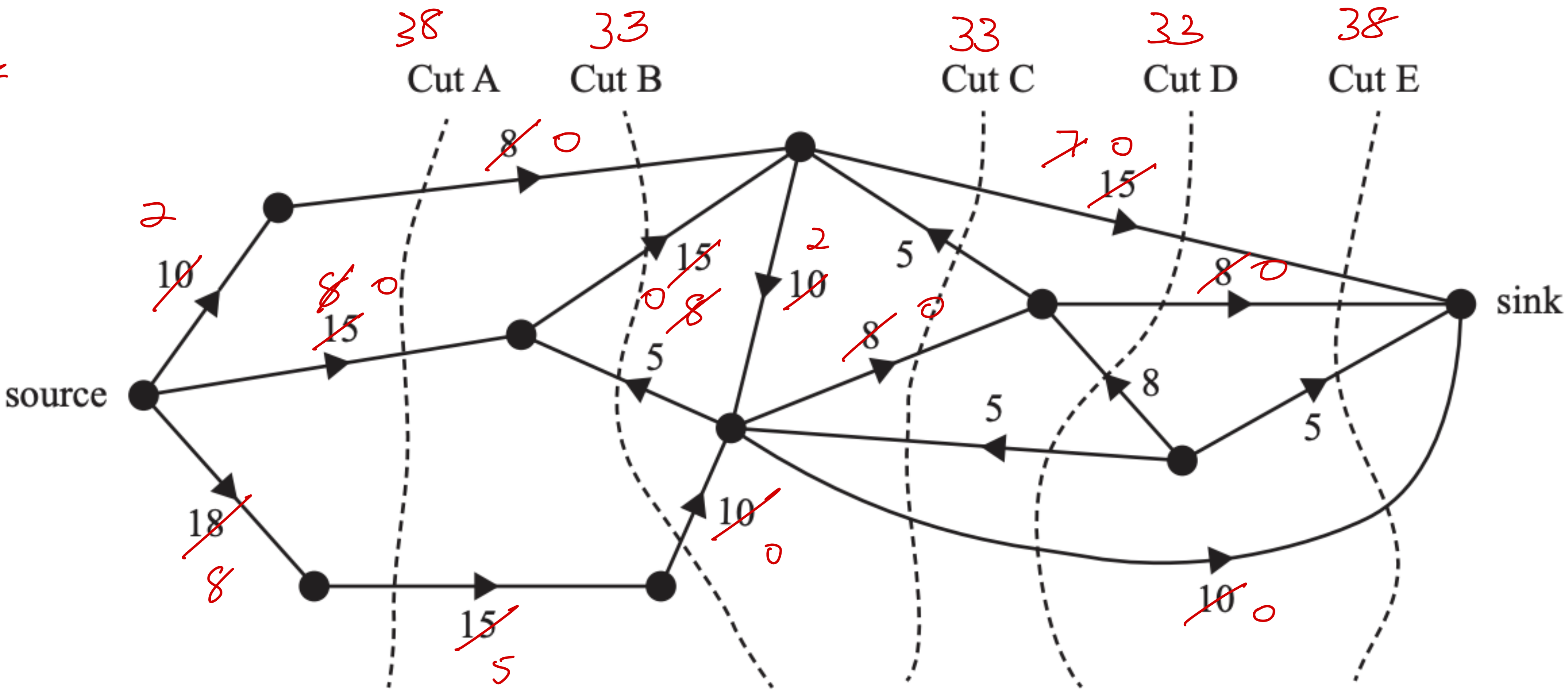
D. 4

E. 5

VCAA Question

This question has been taken from the VCAA 2020 FM Paper 1

8
7 = 33
8
10



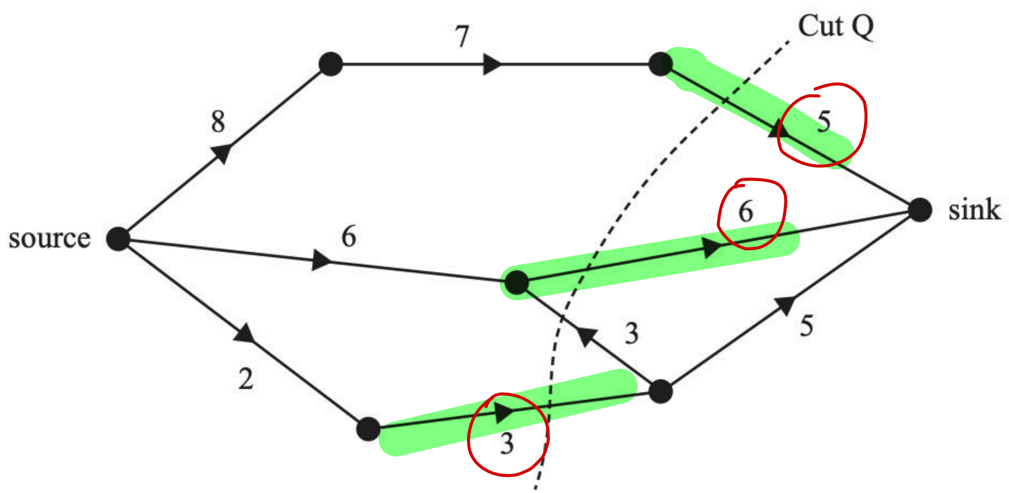
VCAA Question

This question has been taken from the VCAA 2019 FM Paper 1

Question 3

The flow of water through a series of pipes is shown in the network below.

The numbers on the edges show the maximum flow through each pipe in litres per minute.



The capacity of Cut Q, in litres per minute, is

A. 11

B. 13

C. 14

D. 16

E. 17

VCAA Question

The graph below shows the possible number of postal deliveries each day between the Central Mail Depot and the Zenith Post Office.

The unmarked vertices represent other depots in the region.

The weighting of each edge represents the maximum number of deliveries that can be made each day.

This question has been taken from the VCAA 2018 FM Paper 2 Question 1

a. Cut A, shown on the graph, has a capacity of 10.

Two other cuts are labelled as Cut B and Cut C.

i. Write down the capacity of Cut B. 9 — 1 mark

ii. Write down the capacity of Cut C. 13 — 1 mark