



Newton's method for finding solutions

Year 11 and 12
Mathematical Methods

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Learning Objectives

By the end of this lesson, the following topics should have been covered and understood. You should be able to:

- Understand how to use Newton's Methods to find solutions to equations.
- Apply the understanding to a range of problems.



Recap of past learning

In previous videos we have met a number of ways which can be used to find the solutions to equations, including:

- Polynomial Division
- Quadratic Factorisation
- Bisection Method
- Using our CAS (!)

The beauty of Mathematics is that there is always one more way of finding an answer to a particular problem and so this lesson is going to look at another way of finding the roots of polynomial equations.

Note: This content appears in both the Year 11 and Year 12 courses.

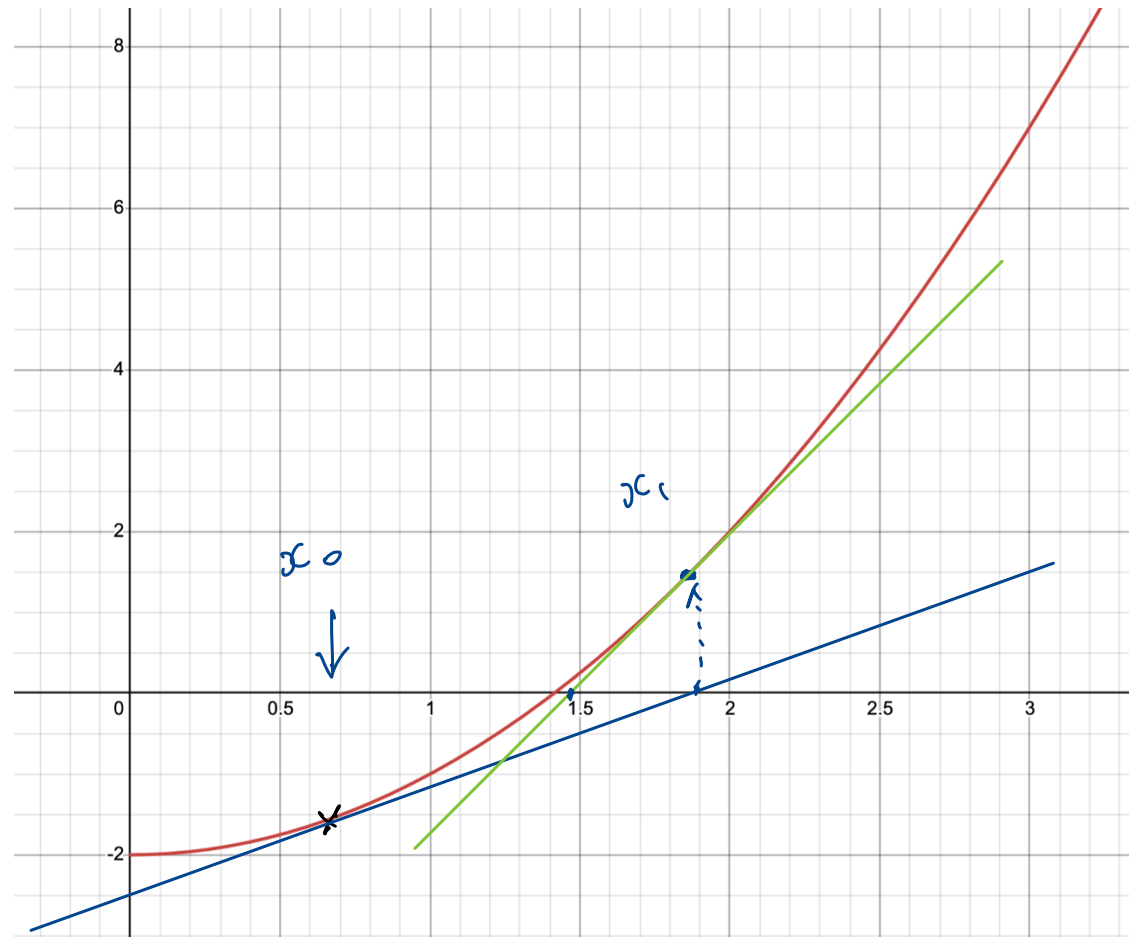


How does Newton's Method work?

Differentiation is about finding the gradient of the tangent to a point.

We know that we can use the gradient of a tangent at a point and the actual point to then find the equation of the tangent.

With this equation I can find where it intercepts the x-axis. The chances are, this will be at a point closer to the root of my equation. Hence, I can start the process again.



The specifics

To find the equation of a tangent we can use the following:

$$y - y_1 = m(x - x_1)$$

We can re-express m at x_1 as $f'(x_1)$

$$y - y_1 = f'(x_1)(x - x_1)$$

If we are looking at finding where this crosses the origin, then we know that $y = 0$ so the equation now becomes:

$$-y_1 = f'(x_1)(x - x_1)$$

And we know that $y_1 = f(x_1)$ giving the final equation as:

$$-f(x_1) = f'(x_1)(x - x_1)$$

This leads nicely to the idea that:

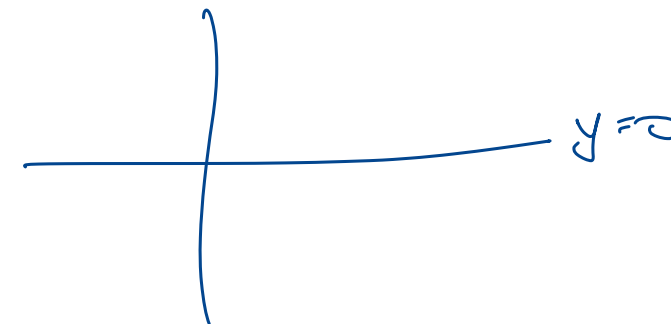
$$\frac{-f(x_1)}{f'(x_1)} = x - x_1$$

Or

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Which means to get from one solution to the next we only need to know the value of the x co-ordinate we are testing

$m(x, y)$



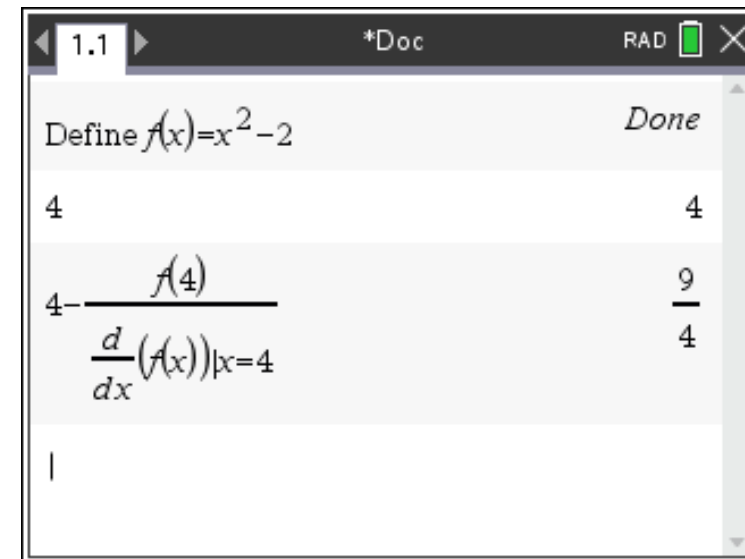
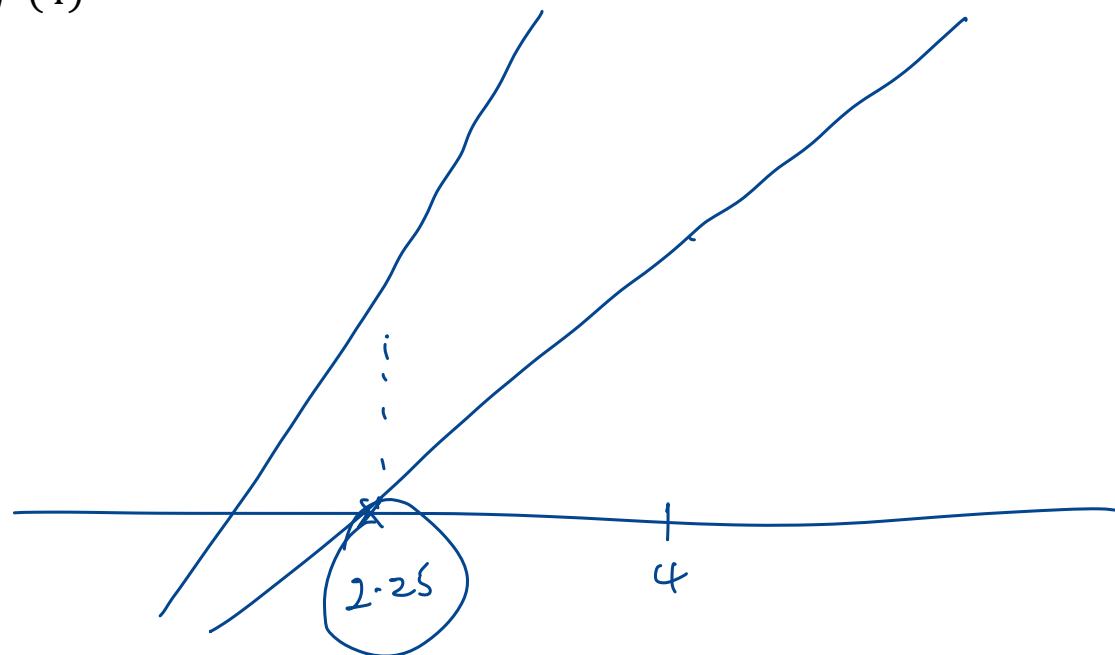
An example

It we look at the example of $y = x^2 - 2$ with a start value of 4.

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x = 4 - \frac{f(4)}{f'(4)}$$

ans



Using the CAS is going to be invaluable here!



An example

It we look at the example of $y = x^2 - 2$ with a start value of 4.

We can now use the iterative nature of the ANS function to provide the next solution

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x = \frac{9}{4} - \frac{f\left(\frac{9}{4}\right)}{f'\left(\frac{9}{4}\right)}$$

$$2.25$$

$$\frac{113}{72}$$

$$1.42189$$

$$1.41423$$

$$1.41421$$

1.1 *Doc RAD

$4 - \frac{f(4)}{\frac{d}{dx}(f(x))|_{x=4}} = 2.25$

$2.25 - \frac{f(2.25)}{\frac{d}{dx}(f(x))|_{x=2.25}} = 1.569444444444$

Using the CAS is going to be invaluable here!



An example

It we look at the example of $y = x^2 - 2$ with a start value of 4.

We can now use the iterative nature of the ANS function to provide the next solution

1.1 *Doc RAD

$$2.25 - \frac{f(2.25)}{\frac{d}{dx}(f(x))|_{x=2.25}} = 1.56944444444$$
$$1.56944444444 - \frac{f(1.56944444444)}{\frac{d}{dx}(f(x))|_{x=1.569444444}} = 1.42189036382$$

1.1 *Doc RAD

$$1.56944444444 - \frac{\frac{d}{dx}(f(x))|_{x=1.569444444}}{\frac{d}{dx}(f(x))|_{x=1.569444444}} = 1.42189036382$$
$$1.4218903638151 - \frac{f(1.4218903638151)}{\frac{d}{dx}(f(x))|_{x=1.421890363}} = 1.41423428594$$

We notice that the values seem to be getting closer and closer to the same x-value.



An example

It we look at the example of $y = x^2 - 2$ with a start value of 4.

We can now use the iterative nature of the ANS function to provide the next solution

n	x_n	$f(x_n)$	$f'(x_n)$
0	4.00000000	14.00000000	8.00000000
1	2.25000000	3.06250000	4.50000000
2	1.56944444	0.46315586	3.13888889
3	1.42189036	0.02177221	2.84378073
4	1.41423429	0.00005862	2.82846857
5	1.41421356	0.00000000	2.82842713
6	1.41421356	0.00000000	2.82842712

$$x_0 - \frac{f(x_0)}{f'(x_0)}$$

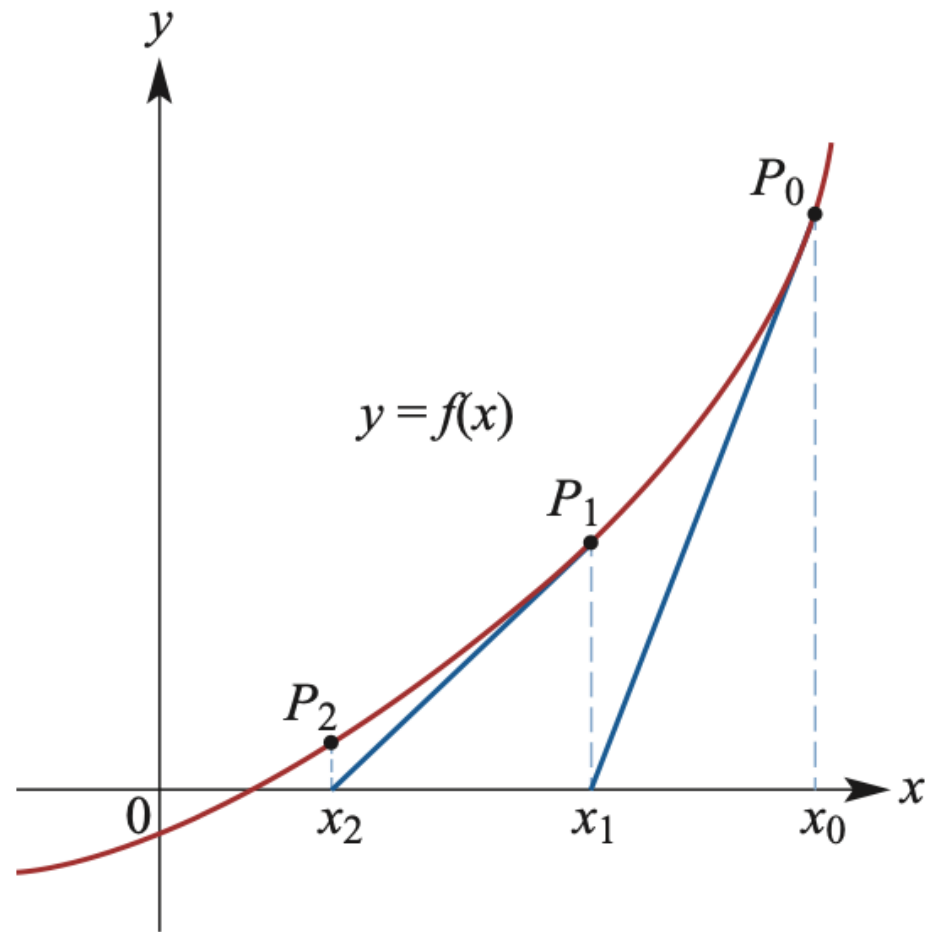
We notice that the values seem to be getting closer and closer to the same x-value.

So, to three decimal places we can see that one of the roots of this equation would be 1.414.



Visually this is what is happening

We can see that the points we are choosing to test are getting closer and closer to the root of the equation.



Example 2

Use Newton's method to solve the equation $-x^3 + 5x^2 - 3x + 4 = 0$ for $x \in [0, \infty)$.

$$\text{ans} = \frac{f(\text{ans})}{\frac{d}{dx}(f(x)) \Big|_{x=\text{ans}}}$$

TI-84 Plus calculator screen showing the function $f(x) = -x^3 + 5x^2 - 3x + 4$ and the first two iterations of Newton's method. The initial guess is 3.8. The first iteration yields a new value of 4.99326923077. The second iteration yields a new value of 4.9932692307692.

$$f(x) = -x^3 + 5x^2 - 3x + 4$$
$$3.8 - \frac{f(3.8)}{\frac{d}{dx}(f(x)) \Big|_{x=3.8}} = 4.99326923077$$
$$4.9932692307692 - \frac{f(4.9932692307692)}{\frac{d}{dx}(f(x)) \Big|_{x=4.9932692307692}} = 4.9932692307692$$

TI-84 Plus calculator screen showing the third iteration of Newton's method. The third iteration yields a new value of 4.6052631578376. The fourth iteration yields a new value of 4.53507147881.

$$4.9932692307692 - \frac{f(4.9932692307692)}{\frac{d}{dx}(f(x)) \Big|_{x=4.9932692307692}} = 4.6052631578376$$
$$4.6052631578376 - \frac{f(4.6052631578376)}{\frac{d}{dx}(f(x)) \Big|_{x=4.6052631578376}} = 4.53507147881$$



Example 2

Use Newton's method to solve the equation $-x^3 + 5x^2 - 3x + 4 = 0$ for $x \in [0, \infty)$.

n	x_n	$f(x_n)$	$f'(x_n)$
0	3.80000000	9.92800000	-8.32000000
1	4.99326923	-10.81199119	-27.86552053
2	4.60526316	-1.44403339	-20.57271468
3	4.53507148	-0.04308844	-19.34990517
4	4.53284468	-0.00004266	-19.31159580
5	4.53284247	0.00000000	-19.31155781
6	4.53284247	0.00000000	-19.31155781

Hence, the solution is $x \approx 4.53284247$

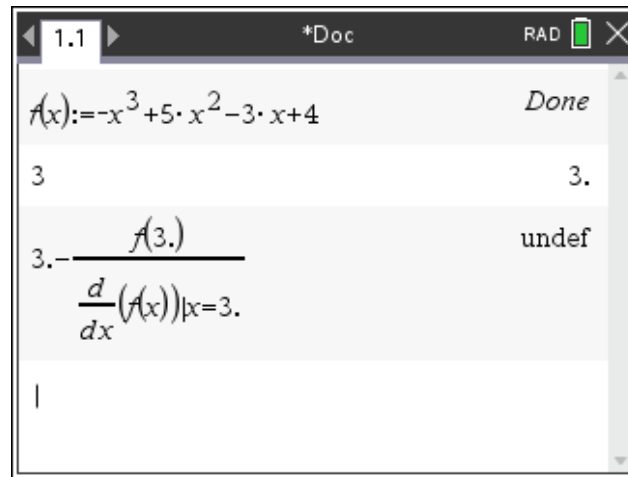


How to choose a starting value

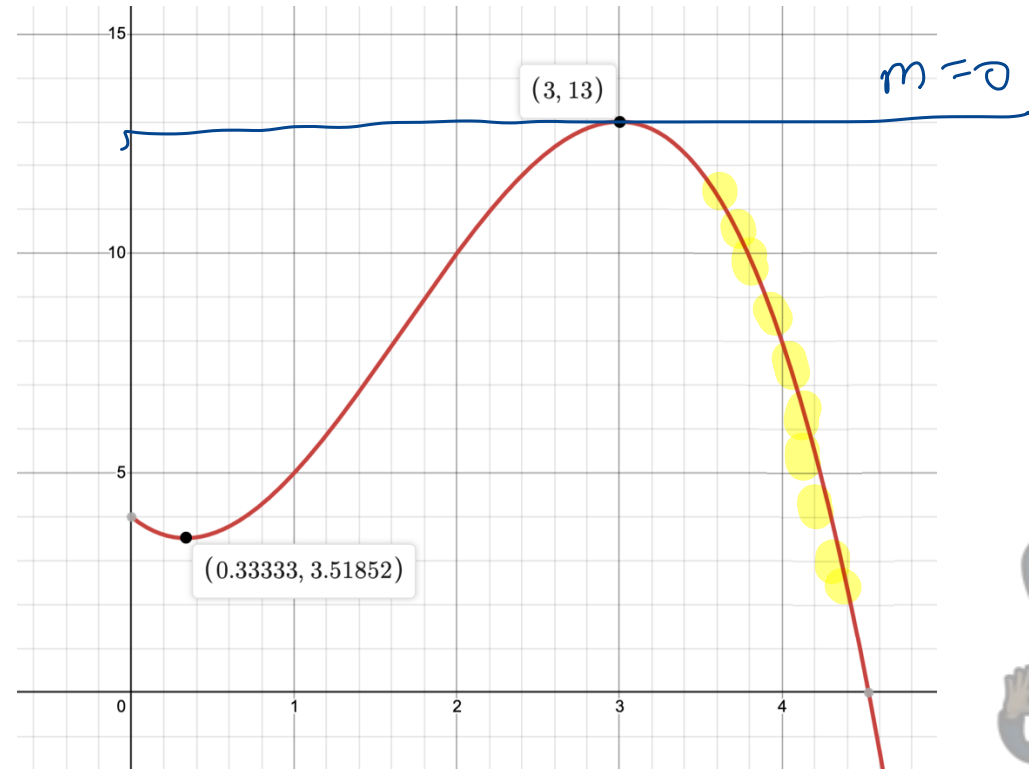
Use Newton's method to solve the equation $-x^3 + 5x^2 - 3x + 4 = 0$ for $x \in [0, \infty)$.

The choice of starting value is quite important.

If we had started with 3 for the above, we would have received the following:



3 is a maximum and hence the tangent will never meet the x-axis.



How to choose a starting value

Use Newton's method to solve the equation $-x^3 + 5x^2 - 3x + 4 = 0$ for $x \in [0, \infty)$.

The choice of starting value is quite important.

If we had started with 2.7 it would have taken over 100 iterations to have arrived at the answer of $x \approx 4.53284247$

Starting with -5 would have taken only 12 iterations.

Thankfully the questions in the textbook give you the starting point!



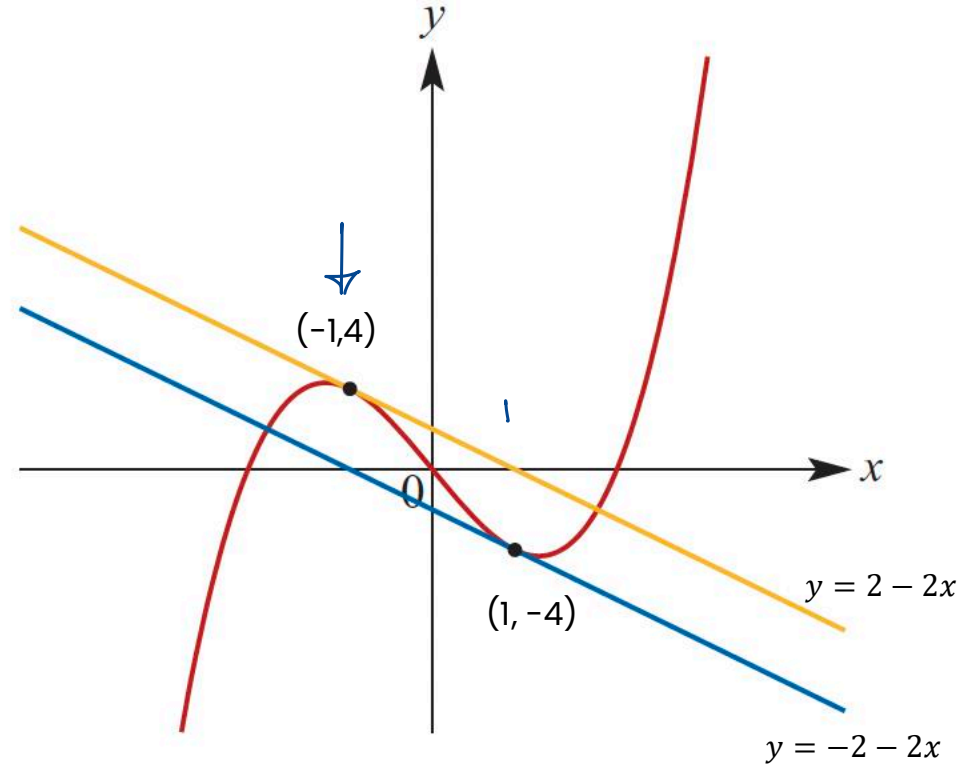
Does it always work?

There are instances where the choice of starting value can lead to unexpected results.

For example, the graph of $y = x^3 - 5x$

Choosing either $x_0 = 1$ or $x_0 = -1$ would create an oscillation and hence never get to any of the roots!

Likewise, we would want to avoid choosing a value which is a maximum or a minimum.



Learning Objectives: Revisited

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