Differentiation: Stationary Points

Year 12
Mathematical Methods

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Learning Objectives

By the end of this lesson, the following topics should have been covered and understood. You should be able to:

- Understand what it means to be a stationary point
- Know how to use differentiation to find the coordinates of stationary points
- Know how to use the CAS to find stationary points
- Apply the learning to a range of different questions

Recap of past learning

We have, in previous lessons, learned how to differentiate functions. We do this to help us find the gradient of the tangent at a point, or instantaneous rates of change, or in kinematics. The applications are numerous and it's important we look at all the cases where we can use differentiation.

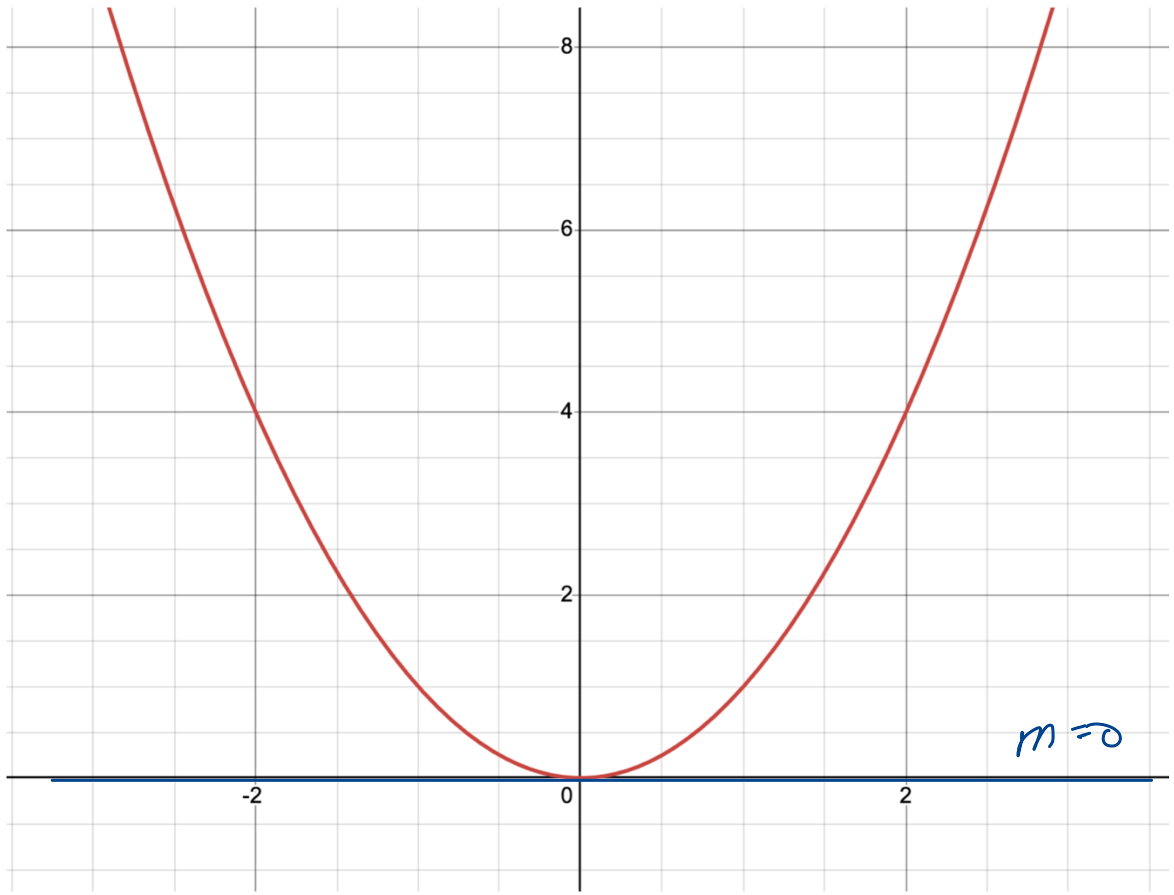
When sketching curves, we know how to find axis intercepts and what the general shape of many functions look like. But how to we find the turning points?

What is a stationary point?

A stationary point is where the gradient of the tangent to the curve at a given point is equal to zero.

Looking at the function on the right, we can see there is a stationary point when x=0. If I were to draw a tangent here, it would be a horizonal line.

We find that maximum and minimum points of graphs will have gradients of zero, but there are other points too which we call

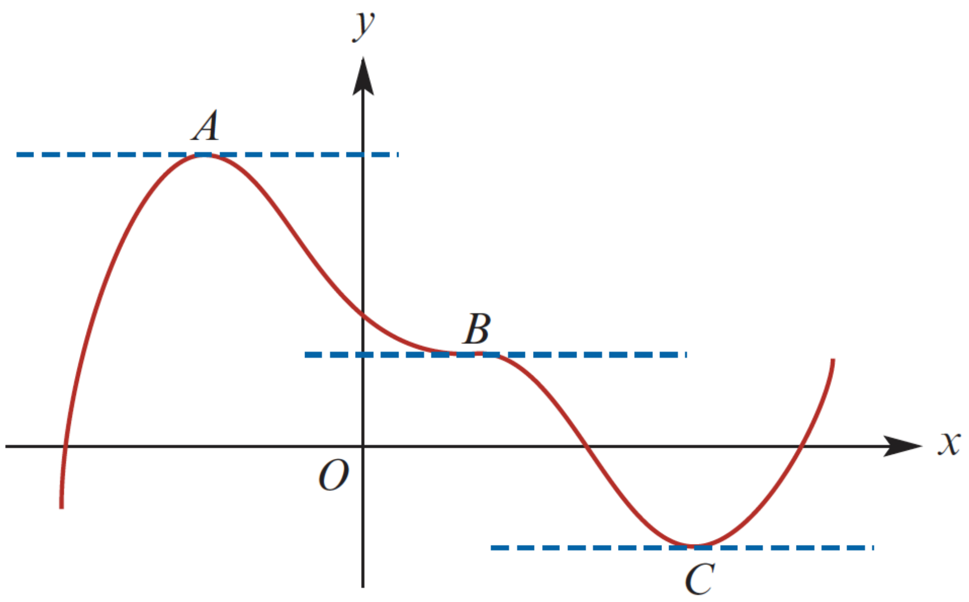


Types of stationary points

We find that maximum and minimum points of graphs will have gradients of zero. We call these called turning points.

There are also points on some graphs called stationary points of inflection where the gradient of the tangent is zero, yet it’s not a maximum or minimum.

A good example is shown on the right.



Finding the coordinates of stationary points

If we know that the gradient of the tangent to the line at a stationary point is zero, we can simply differentiate the function and solve for any points where the differential is equal to zero!

$y = f(x)$

$f'(x) = \underline{\underline{0}}$

Example

Find the stationary points of the following functions:

a $y = \cancel{9} + 12x - 2x^2$ **b** $y = 4 + 3x - x^3$ **c** $p = 2t^3 - 5t^2 - 4t + 13,\ t > 0$

a. $y' = 12 - 4x$

$\therefore$ SP $y' = 0$

$12 - 4x = 0$

$-4x = -12$

$x = \underline{\underline{3}}$

$y = 27$

$\therefore$ SP $= \underline{\underline{(3, 27)}}$

Example

Find the stationary points of the following functions:

a $y = 9 + 12x - 2x^2$ **b** $y = \cancel{4} + 3x - x^3$ **c** $p = 2t^3 - 5t^2 - 4t + 13,\ t > 0$

$\therefore$ SP $(1,6)$ & $(-1,2)$

$y' = 3 - 3x^2$

@ SP $y' = 0$

$3 - 3x^2 = 0$

$-3x^2 = -3$

$x^2 = 1$

$x = \pm 1$

$\therefore x = 1$ $\quad x = -1$

$y = 6$ $\quad y = 2$

Example

Find the stationary points of the following functions:

a $y = 9 + 12x - 2x^2$ **b** $y = 4 + 3x - x^3$ **c** $p = 2t^3 - 5t^2 - 4t + 13,\ t > 0$

-6

$1 \mid -6$

$p' = 6t^2 - 10t - 4$

@ SP $p' = 0$

$6t^2 - 10t - 4 = 0$

$3t^2 - 5t - 2 = 0$

$3t^2 - 6t + t - 2 = 0$

$3t(t-2) + 1(t-2) = 0$

$(3t+1)(t-2) = 0$

$3t + 1 = 0 \qquad t - 2 = 0$

$t = -1/3 \qquad t = 2$

$\cancel{t = -\frac{1}{3}}$ $\quad t = 2$

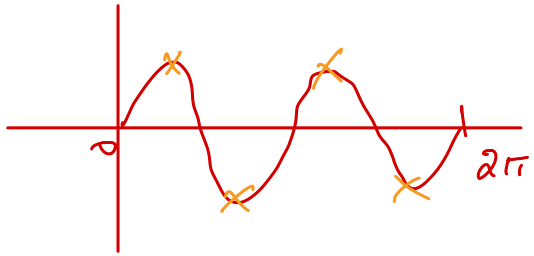
$\cancel{p =}$ $\quad p = 1$

$\therefore SP = \underline{(2, 1)}$

Example

Find the stationary points of the following functions:

a $y = \sin(2x),\ x \in [0, 2\pi]$ **b** $y = e^{2x} - x$ **c** $y = x\log_e(2x),\ x \in (0, \infty)$



a. $y = \sin 2x$

$y = \sin(u) \qquad u = 2x$

$\frac{dy}{du} = \cos u \qquad \frac{du}{dx} = 2$

$\frac{dy}{dx} = 2\cos 2x$

@ SP $\frac{dy}{dx} = 0$

$2\cos 2x = 0$

$x = \frac{\pi}{4} \quad \frac{3\pi}{4} \quad \frac{5\pi}{4} \quad \frac{7\pi}{4}$

$y = 1 \quad -1 \quad 1 \quad -1$

Example

Find the stationary points of the following functions:

a $y = \sin(2x),\ x \in [0, 2\pi]$ **b** $y = e^{2x} - x$ **c** $y = x\log_e(2x),\ x \in (0, \infty)$

$y = e^{2x} - x$

$y' = 2e^{2x} - 1$

sp @ $y' = 0$

$\therefore\ 2e^{2x} - 1 = 0$

$2e^{2x} = 1$

$e^{2x} = 1/2$

$x = \frac{-\ln 2}{2}$

$y = \frac{\ln 2}{2} + \frac{1}{2}$

Example

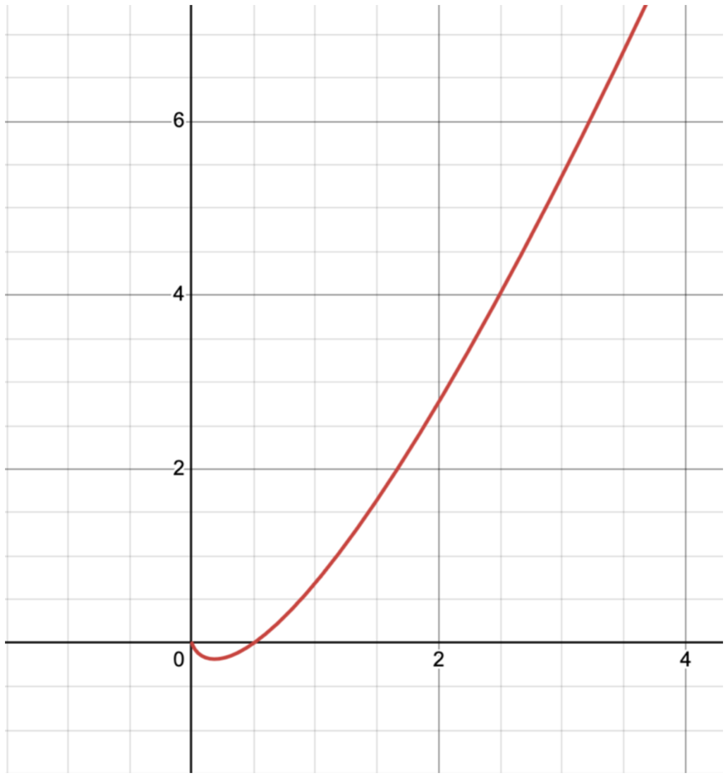
Find the stationary points of the following functions:

a $y = \sin(2x)$, $x \in [0, 2\pi]$ **b** $y = e^{2x} - x$ **c** $y = x\log_e(2x)$, $x \in (0, \infty)$

$y = x \cdot \log_e 2x$

$x = \dfrac{e^{-1}}{2} = \underline{\underline{\dfrac{1}{2e}}}$

$y = \underline{\underline{\dfrac{-1}{2e}}}$



Remember: a sketch can save hours of work!

Example: Using the CAS

The curve with equation $y = x^3 + ax^2 + bx + c$ passes through the point (0, 5) and has a stationary point at (2, 7). Find a, b and c.

$n(0) = 5$

$n(2) = 7$

$y' = 3x^2 + 2ax + b$

$0 = 3(2)^2 + 2a.2 + b$

$0 = 12 + 4a + b$

$\therefore a = -\frac{9}{2}$ $\quad b = 6$ $\quad c = 5$

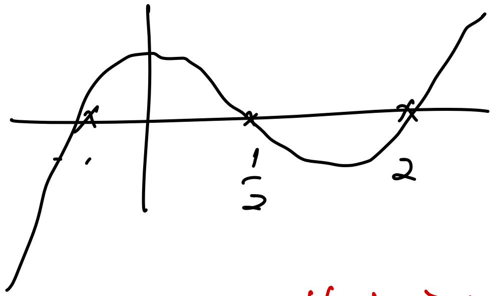
VCAA Past Paper Question: 2024 Paper 2 Question 1

c. Let g be the function $g : R \to R, g(x) = (x+1)^2(x-2)^2$, which is the function f where $a = 1$.

i. Find $g'(x)$. 1 mark

ii. Find the coordinates of the local maximum of g. 1 mark

iii. Find the values of x for which $g'(x) > 0$. 1 mark



$g'(x) > 0$

↑

above x-axis

(i) $g'(x) = 2(x-2)(x+1)(2x-1)$ ✓

(ii) max $= \left(\frac{1}{2}, \frac{81}{16}\right)$ ✓

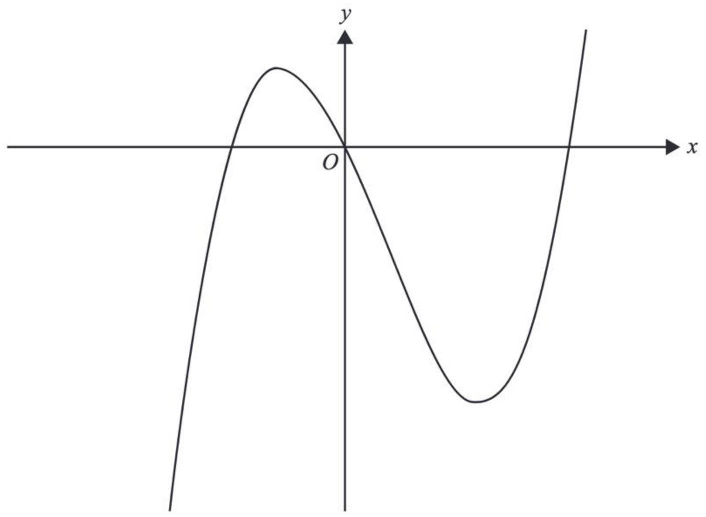
(iii)

$\therefore x \in \left(-1, \frac{1}{2}\right) \cup (2, \infty)$

VCAA Past Paper Question: 2023 Paper 2 Question 1

Question 1 (11 marks)

Let $f: R \to R$, $f(x) = x(x-2)(x+1)$. Part of the graph of f is shown below.



b. Find the coordinates of the stationary points of f. 2 marks

$f(x) = x(x-2)(x+1)$

$f'(x) = 3x^2 - 2x - 2$

sp @ $f'(x) = 0$

$3x^2 - 2x - 2 = 0$

$x = \dfrac{\sqrt{7}+1}{3}$ or $x = \dfrac{1-\sqrt{7}}{3}$

$y = \dfrac{-20 - 14\sqrt{7}}{27}$ $y = \dfrac{-20 + 14\sqrt{7}}{27}$

Learning Objectives: Revisited

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