Rates of change

Year 12

Mathematical Methods

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Learning Objectives

By the end of this lesson, the following topics should have been covered and understood. You should be able to:

- Understand the what it means to be an “average rate of change” and “instantaneous rate of change”
- Be able to find average and instantaneous rates of changes
- Be able to apply the learning to a number of different questions

Recap of past learning

This is the second lesson in this series looking at applications of differentiation.

Rates of change are really important and appear in exam and SAC questions all the time!

It’s important to remember that the gradient of a line can be found using:

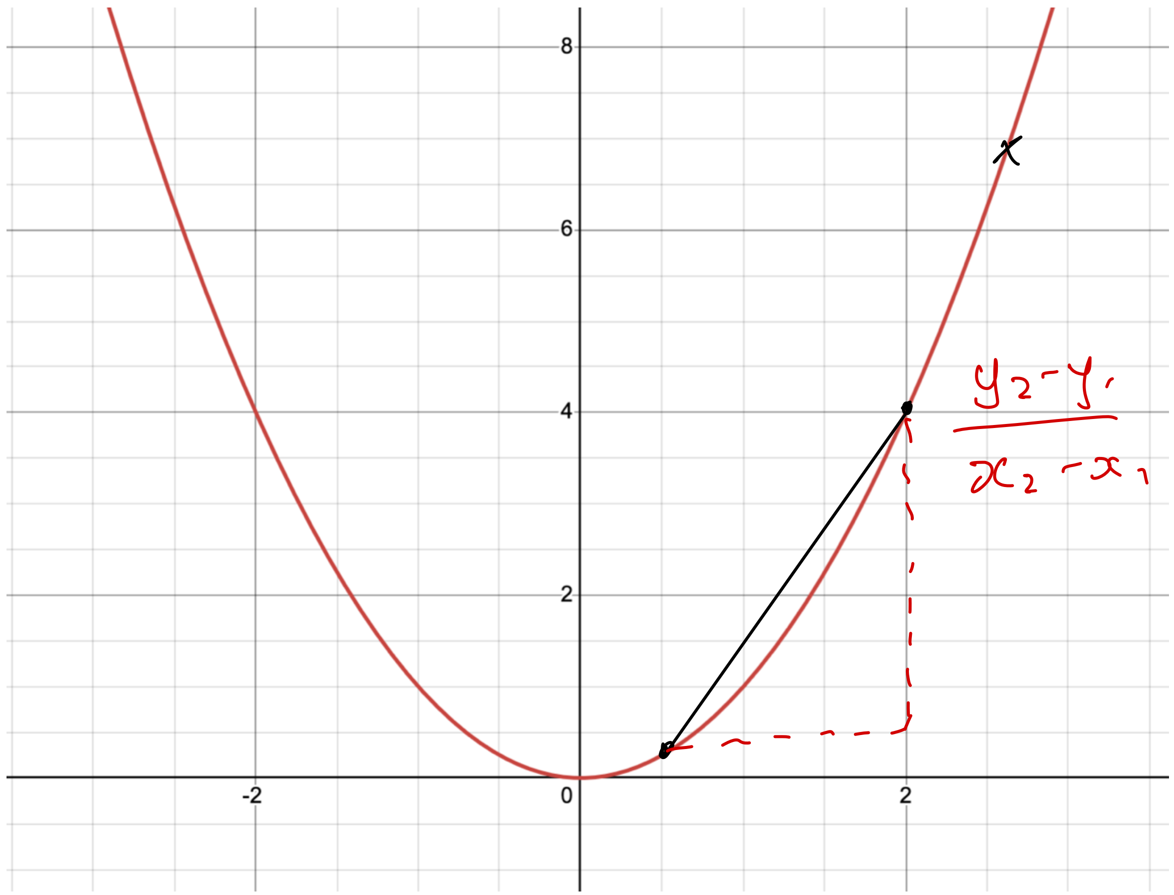
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

What is a a rate of change?

If I join two points together and find the gradient of the line connecting those two points, I am finding the **average rate of change**.

If I want to find the **instantaneous rate of change**, I need to find the gradient of the tangent at that one point.

Thankfully, I can use differentiation to help me find instantaneous rates of change!



Example

For the function with rule $f(x) = x^2 + 2x$, find:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

Remember: This is a CAS course and, as such, we can use the CAS to help (or check!).

a. $x_1 = 2 \qquad f(x) = y_1 = \underline{\underline{8}}$

$x_2 = 3 \qquad f(x) = y_2 = 15$

$\therefore \text{ ave rate } = \dfrac{15-8}{3-2} = \underline{\underline{7}}$

b. $x = 2 \qquad y = \underline{\underline{8}}$

$x = 2 + h \qquad y = (2+h)^2 + 2(2+h)$

$= 4 + 4h + h^2 + 4 + 2h$

$= 8 + 6h + h^2$

$\therefore \text{ arc } = \dfrac{h^2 + 6h + \cancel{8} - \cancel{8}}{\cancel{2} + h - \cancel{2}}$

$= \dfrac{h^{\cancel{2}} + 6\cancel{h}}{\cancel{h}}$

$= \underline{\underline{h + 6}}$

c. $f'(x) = 2x + 2 \qquad f'(2) = \underline{\underline{6}}$

Example

A balloon develops a microscopic leak and gradually decreases in volume. Its volume, V cm^3, at time t seconds is $V = 600 - 10t - \frac{1}{100}t^2$, $t \geq 0$.

a Find the rate of change of volume after:

i 10 seconds **ii** 20 seconds

b For how long could the model be valid?

a. i) $\frac{dV}{dt} = -10 - \frac{2}{100}.t$

$\frac{dV}{dt} = -10 - \frac{2.1\cancel{0}}{10\cancel{0}}$

$= -10 - \frac{2}{10}$

$= -10\frac{1}{5}$ cm^3/s

a ii) $\frac{dV}{dt} = -10\frac{2}{5}$ cm^3/s

b. $0 = 600 - 10t - \frac{1}{100}t^2$

$t = 100(\sqrt{31} - 5)$

$0 \leq t \leq 100(\sqrt{31} - 5)$

Example

A pot of liquid is put on the stove. When the temperature of the liquid reaches 80°C, the pot is taken off the stove and placed on the kitchen bench. The temperature in the kitchen is 20°C. The temperature of the liquid, T°C, at time t minutes is given by

$$T = 20 + 60e^{-0.3t}$$

a Find the rate of change of temperature with respect to time in terms of T.

b Find the rate of change of temperature with respect to time when:

i $T = 80$ **ii** $T = 30$

$$T = 20 + 60.e^{-0.3t}$$

$$T - 20 = 60.e^{-0.3t}$$

$$\frac{T-20}{60} = e^{-0.3t}$$

a. $\dfrac{dT}{dt} = 60(-0.3).e^{-0.3t}$

$= -18e^{-0.3t} = -18.\left(\dfrac{T-20}{60}\right)$

b. i. $\dfrac{dT}{dt} = -18\left(\dfrac{80-20}{60}\right) = -18\,°C/min$

ii. $\dfrac{dT}{dt} = -18\left(\dfrac{30-20}{60}\right) = -18.\dfrac{1}{6} = -3\,°C/min.$

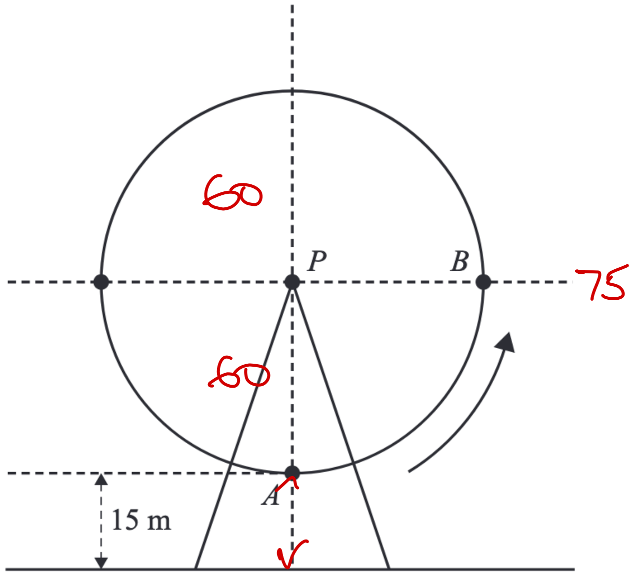
VCAA Past Paper Question: 2023 Paper 2

Question 2 (11 marks)

The following diagram represents an observation wheel, with its centre at point P. Passengers are seated in pods, which are carried around as the wheel turns. The wheel moves anticlockwise with constant speed and completes one full rotation every 30 minutes. When a pod is at the lowest point of the wheel (point A), it is 15 metres above the ground. The wheel has a radius of 60 metres.

Consider the function $h(t) = -60\cos(bt) + c$ for some $b, c \in R$, which models the height above the ground of a pod originally situated at point A, after time t minutes.

c. Find the average rate of change, in metres per minute, of the height of a pod on the wheel as it travels from point A to point B. 1 mark



$$h(t) = -60.\cos\left(\frac{\pi}{15}.t\right) + 75$$

$t = 0 \quad h(0) = 15$

$t = 7.5 \quad h(7.5) = 75$

$$\therefore \text{ave rate chng} = \frac{75 - 15}{7.5 - 0} = \underline{8 \text{ m/min}}$$

Note: The following was a previous part of the question

Show that $b = \dfrac{\pi}{15}$ and $c = 75$.

Learning Objectives: Revisited

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