Exponential growth and decay

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By the end of the teaching of this module I would hope that the following has been covered and understood. Hopefully you will be able to apply the learning to any exam questions:

- Know how to use exponentials and logarithms to solve "Real World Problems"
- Know how to identify which exponential formula to apply to a certain question
- Understand what it means by:
  - Cell growth
  - Radioactive Decay
  - Population Growth

RECAP

In previous lessons we have been dealing with using how to use exponentials and logarithms to manipulate and solve problems. All problems, until now, have been "standard maths problems". This lesson will now look at how to extract information from questions and apply the appropriate exponential equation to solve

Case 1: Cell growth

We are all aware of how cells divide.
In many cases, they divide in half every time period.
When dealing with this case, we can use the following exponential formula:

$$N = N_0 \times 2^{\frac{t}{T_D}}$$

N = Number of cells
t= number of **minutes** elapsed since the start
T_D=generation time in **minutes** (the time it takes to split)

Example:
The following example has been taken, with permission, from the Cambridge Mathematical Methods Units 3 and 4 Textbook

What is the generation time of a bacterial population that increases from 5000 cells to 100000 cells in four hours of growth?

T_D

$N = N_0 \times 2^{t/T_D}$

$4 \times 60 = 240$

$100\,000 = 5000 \times 2^{\frac{240}{T_D}}$

$20 = 2^{240/T_D}$

$\log_2 20 = \frac{240}{T_D}$

$T_D = \frac{240}{\log_2 20}$

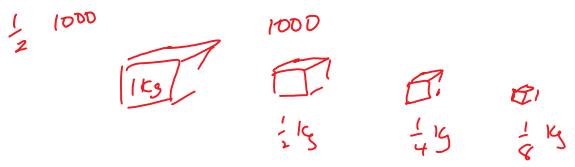
$= 55.53$ mins

Case 2: Radioactive decay

When dealing with radioactivity we are looking at the concept of half-lives.
A half-life is the time it takes for half of a radioactive substance to decay.
When trying to solve these problems we use the follow equation:

$$A = A_0 e^{-kt}$$

A is the amount of material left
t is the time taken since the start
A_0 is the initial amount of material
k is constant of decay



Example:
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After 1000 years, a sample of radium-226 has decayed to 64.7% of its original mass. Find the half-life of radium-226.

$t = 1000$

A_0

$A = 0.647 \times A_0$

$A = A_0 e^{-kt}$

$0.647A_0 = A_0 e^{-k \cdot 1000}$

$0.647 = e^{-1000k}$

$k = 0.000435408945\ldots$

$k \approx 0.000435$

$\frac{1}{2}A_0 = A_0 e^{-1kt}$

$\frac{1}{2} = e^{-kt}$

$\frac{1}{2} = e^{-0.000435..\times t}$

$t = 1592$ years

Case 3: Population growth

The final case relates to the idea of population growth.
We can use the same formula for radioactive decay (so long as we know) with a positive constant.
So long as we know there is growth happening we can use the below formula:

$$A = A_0 e^{kt}$$

A is the amount at time t
t is the time taken since the start
A_0 is the initial amount
k is a growth constant

Example:
The following example has been taken, with permission, from the Cambridge Mathematical Methods Units 3 and 4 Textbook

The population of a town was 8000 at the beginning of 2007 and 15000 at the end of 2014. Assume that the growth is exponential.

- Find the population at the end of 2016.
- In what year will the population be double that of 2014?

$A = A_0 e^{kt}$

$15000 = 8000 \times e^{k \cdot 8}$

$k = 0.0785760..$

$A = 8000 \times e^{(0.07857 \times 10)}$

$= 17552$ people

$30000 = 8000 \times e^{(0.07857.. \times t)}$

$x = 16.82$

$x = 7$

2023

Harder example:
This example below isn't exponential growth and looks more at the idea of percentage growth.
The key ideas relate to the maths for compound interest.
A great example is shown below:

Example:
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There are approximately ten times as many red kangaroos as grey kangaroos in a certain area. If the population of grey kangaroos increases at a rate of 11% per annum while that of the red kangaroos decreases at 5% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

Start: $R_0 = 10G_0$ End $G = 10R$

$G = G_0(1.11)^n$ $R = R_0(0.95)^n$

$G = 10R$

$\frac{R_0}{G_0}(1.11)^n(0.95)^n = 10 \times$

$(1.11)^n = 10 \times 10 \times (0.95)^n$

$(1.11)^n = 100 \times (0.95)^n$

$n = 29.59$ years