Position, Velocity and acceleration

Year 12 Specialist Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what each of the following terms mean:
  - Position
  - Displacement
  - Distance
  - Velocity
  - Speed
  - Average velocity
  - Instantaneous velocity
  - Acceleration.
- How to use differentiation and anti-differentiation to find positions, velocities and accelerations.

Recap of past learning

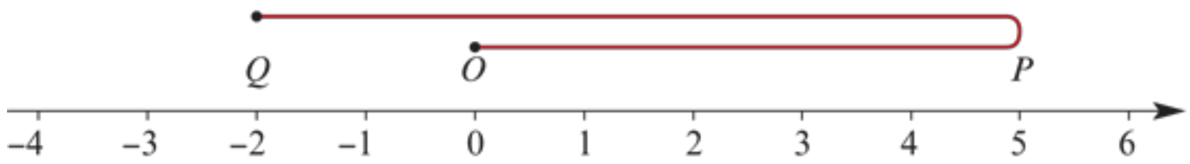
This is another new topic which builds a little on the work covered in Methods 1 and 2, Methods 3 and 4 and a whole lot oh Physics! As this is a new topic, there isn't much to recap from the learning.

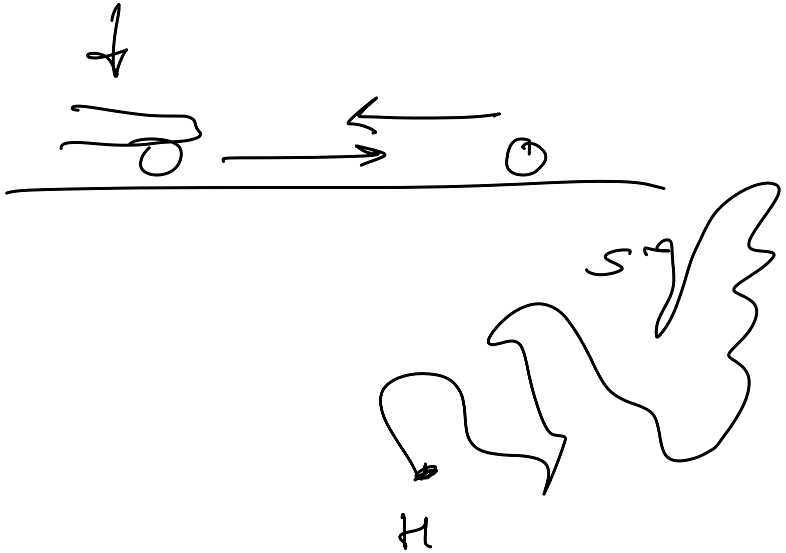
It is important to ensure that you have a complete understanding of how to differentiate and integrate.

I know, when I was at school, this topic (and those which were related) took one-third of my whole A-Level, so we're going to merely scratch the surface of this topic. We will do enough to be able to complete a SAC though! So, it's important to have complete understanding.

Position, Distance and Displacement

The primary thing to note with Kinematics, is the language and the subtle difference between the terms





The **position** of the object is given with respect to a **fixed origin**. It has a sign which denotes which side of the origin it sits. Positive is generally meant to be on the right of the origin.; negative to the left.

The displacement is meant to be the change in position of the particle over the course of a journey. Again, this is taken with respect to the origin. In the above example the displacement would be -2 units.

The distance travelled in the above example would be 12 units.

Note: It's really hard to define the word distance without using the word distance.

Velocity and Speed

Velocity is a vector and hence must make reference to a direction of travel. We would do this using a positive and negative sign.

Speed is a scalar quantity.

Instantaneous velocity
This is the velocity at a point. It is the gradient of the tangent to the point of a distance time graph. Hence we can find the velocity at a point using:

$$v = \frac{dx}{dt}$$

Average velocity
This is the gradient of the chord joining two points on a distance time graph.
We can find the gradient of this chord using:

$$average\ velocity = \frac{x_2 - x_1}{t_2 - t_1}$$

Note: It's important to sketch the functions first and then find points of intersection. Using the CAS is the quickest way!

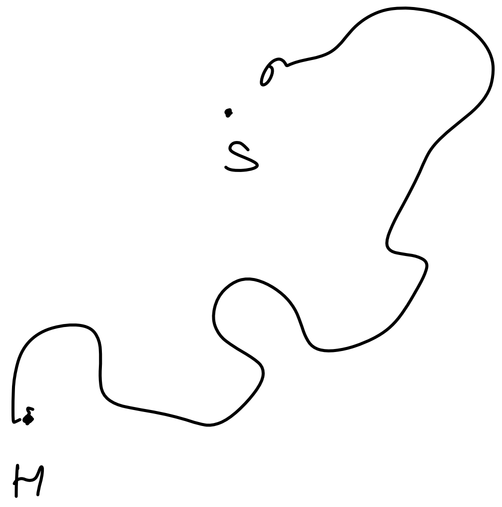
Speed and average speed

A reminder that speed is the magnitude of the velocity.

Average speed is given by the following formula:

$$Average\ speed = \frac{total\ distance\ travelled}{total\ time\ taken}$$

Note: It's important to sketch the functions first and then find points of intersection. Using the CAS is the quickest way!



Acceleration

This is the rate of change of velocity with respect to time.
We have met this before and hence we know that the **instantaneous acceleration** can be found by:

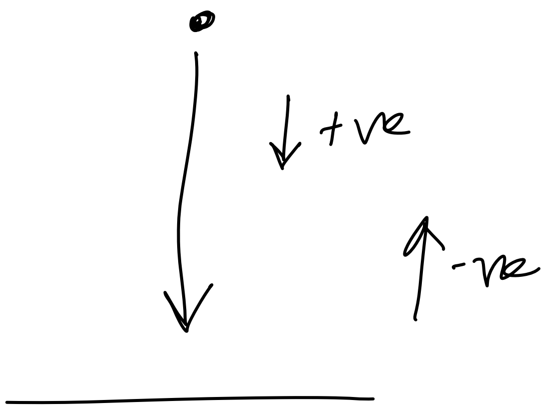
$$a = \frac{dv}{dt}$$

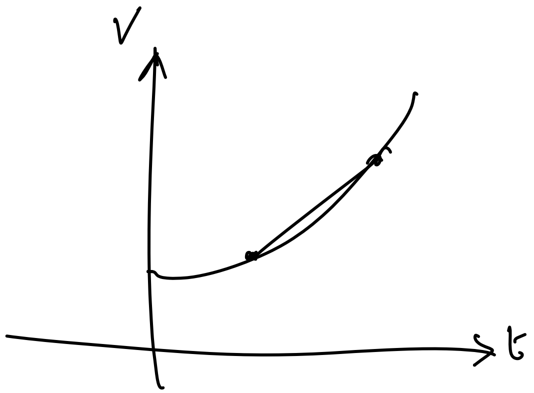
This is the gradient of a tangent to the point of interest of a velocity time graph.

Average acceleration:
It makes sense that the average acceleration hence is:

$$average\ acceleration = \frac{v_2 - v_1}{t_2 - t_1}$$

Note: It's important to sketch the functions first and then find points of intersection. Using the CAS is the quickest way!





Example

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 7t + 6, t \geq 0$.

- Find its initial position.
- Find its position at t=4.

$t = 0$

$x = t^2 - 7t + 6$

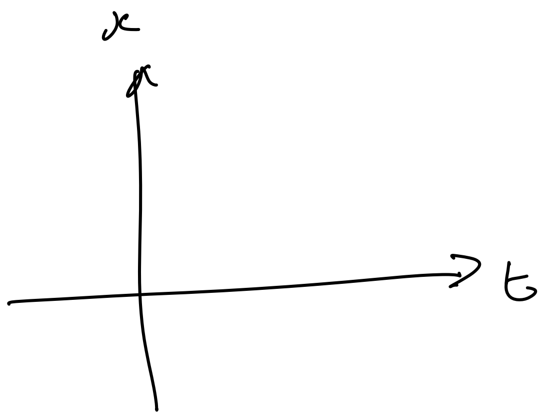
$x = 6$ cm

$t = 4$

$x = 4^2 - 7.4 + 6$

$= 16 - 28 + 6$

$= -6$ cm.



Example

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = 3t - t^3, for\ t \geq 0$. Find:

- its initial position
- its position when $t = 2$
- its initial velocity
- its velocity when $t = 2$
- its speed when $t = 2$
- when and where the velocity is zero.

Note: You should always draw a sketch of the functions to see if the particle comes back on itself!

- $t = 0 \quad \underline{\underline{x = 0}}$

$t = 2 \quad 3 \cdot 2 - 2^3$

$= 6 - 8$

$= \underline{\underline{-2\text{ cm}}}.$

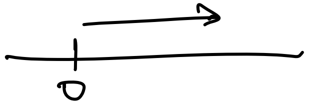
$v = \frac{dx}{dt} = \underline{\underline{3 - 3t^2}}$

$t = 0 \quad v = \underline{\underline{3}}\text{ cm/s}$

$v_2 = 3 - 3 \cdot 2^2$

$= 3 - 12$

$= -9\text{ cm/s}$



- $\text{speed} = \underline{\underline{9}}\text{ cm/s}$

$v = 3 - 3t^2$

$0 = 3 - 3t^2$

$3t^2 = 3$

$t^2 = 1$

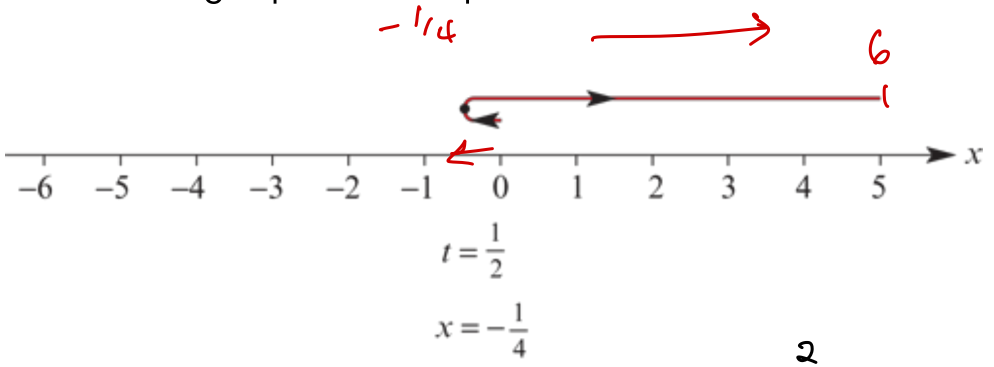
$t = \underline{\underline{\pm 1}}\text{ sec}$

$\therefore t = \underline{\underline{1}}$

Example

The motion of a particle moving along a straight line is defined by $x(t) = t^2 - t$, where x m is the position of the particle relative to O at time t seconds $(t \geq 0)$. Find:

- the average velocity of the particle in the first 3 seconds
- the distance travelled by the particle in the first 3 seconds
- the average speed of the particle in the first 3 seconds.



$x = t^2 - t$

$v = 2t - 1$

$t = 0$ | $t = 3$

$v = -1$ | $v = 5$

$(0, -1)$ | $(3, 5)$

$\text{ave vel} = \frac{5-(-1)}{3-0} = \frac{6}{3} = 2 \text{ m/s}$

$AS = \frac{TDT}{TTT} = \frac{6\frac{1}{2}}{3} = \frac{13}{6} \text{ m/s}$

$v = 2t - 1$

$0 = 2t - 1$

$t = \frac{1}{2} \text{ sec}$

$x = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}$

$x = 3^2 - 3 = 6$

$\therefore TDT = \frac{1}{4} + \frac{1}{4} + 6 = 6\frac{1}{2} \text{ m}$

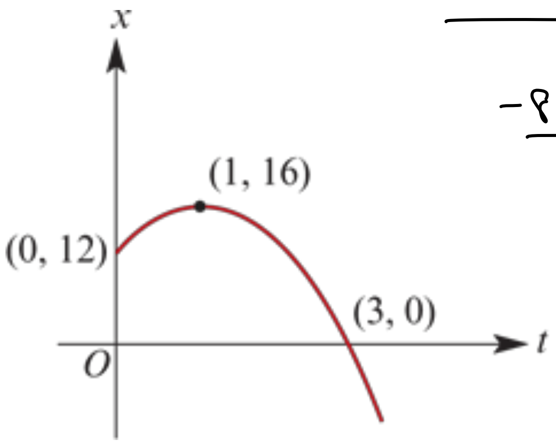
Example

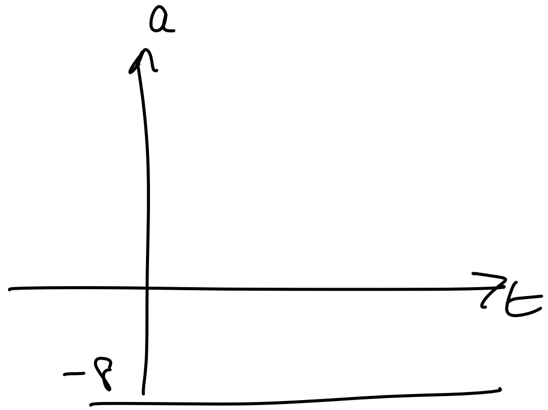
An object travelling in a horizontal line has position x metres, relative to an origin O, at time t seconds, where $x = -4t^2 + 8t + 12, t \geq 0$.

- Sketch the position–time graph, showing key features.
- Find the velocity at time t seconds and sketch the velocity–time graph.
- Find the acceleration at time t seconds and sketch the acceleration–time graph.
- Represent the motion of the object on a number line.
- Find the displacement of the object in the third second.
- Find the distance travelled in the first 3 seconds.

$v = -8t + 8$ m/s

$a = -8$ m/s^2



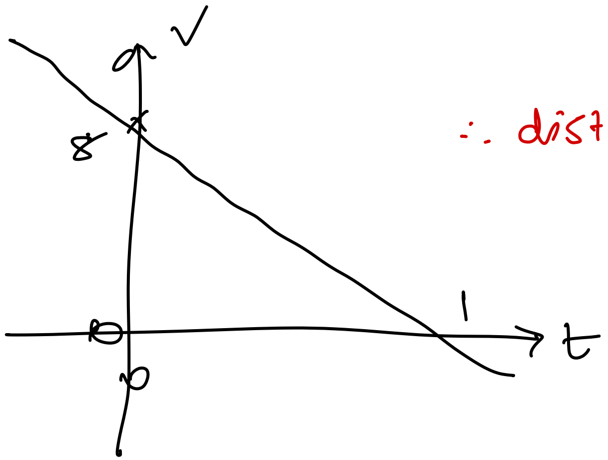


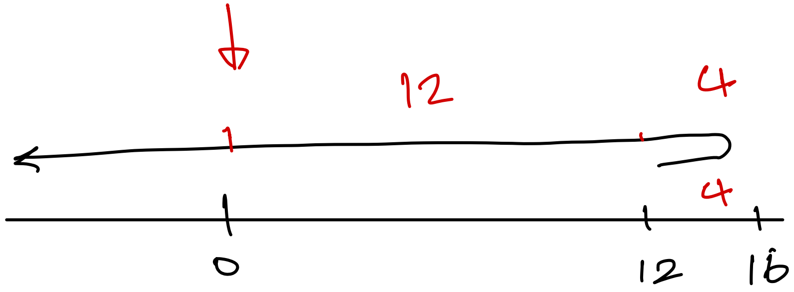
$x_2 = 12$

$x_3 = 0$

$\therefore$ disp $= -12$ m

$\therefore$ dist $= 20$ m

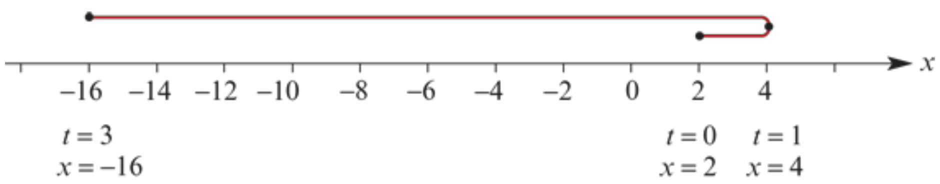




Example

An object moves in a horizontal line such that its position, x m, relative to a fixed point at time t seconds is given by $x = -t^3 + 3t + 2, t \geq 0$. Find:

- when the position is zero, and the velocity and acceleration at that time
- when the velocity is zero, and the position and acceleration at that time
- when the acceleration is zero, and the position and velocity at that time
- the distance travelled in the first 3 seconds.



$\therefore d_t = \underline{\underline{22m}}$

- $x = -t^3 + 3t + 2$

$v = -3t^2 + 3$

$a = \underline{\underline{-6t}}$

$\therefore t = \underline{\underline{2m}}$ $\quad v = \underline{\underline{-9m/s}}$

$a = \underline{\underline{-12m/s^2}}$

- $0 = -3t^2 + 3$

$-3 = -3t^2$

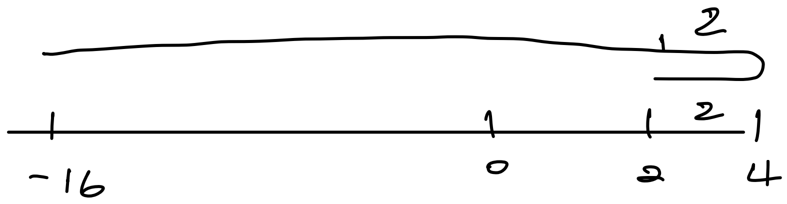
$t^2 = 1$

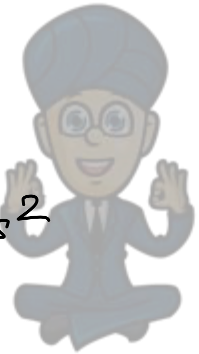
$t = \underline{\underline{1 \text{ sec}}}$

$x = \underline{\underline{4m}}$ $\quad a = \underline{\underline{-6m/s^2}}$

$a = 0 \ @ \ t = 0$

$x = \underline{\underline{2m}}$ $\quad v = 3\underline{\underline{m/s}}$





Example – Using antidifferentiation

The acceleration of a particle moving in a straight line, in m/s^2, is given by

$$\frac{d^2y}{dt^2} = \cos(\pi t)$$

at time t seconds. The particle's initial velocity is 3 m/s and its initial position is $y = 6$. Find its position, y m, at time t seconds.

$\therefore y = -\frac{1}{\pi^2} \cdot \cos(\pi t) + 3t + 6 + \frac{1}{\pi^2}$

x diff v int a

$\therefore v = \int \cos(\pi t) . dt$ $\quad t = 0, v = 3$

$v = \frac{1}{\pi} \sin(\pi t) + c$ $\quad \therefore v = \frac{1}{\pi} \sin(\pi t) + 3$

$3 = \frac{1}{\pi} . \sin(0) + c$

$\therefore x = \int \left(\frac{1}{\pi} \sin(\pi t) + 3\right) dt$

$c = 3$

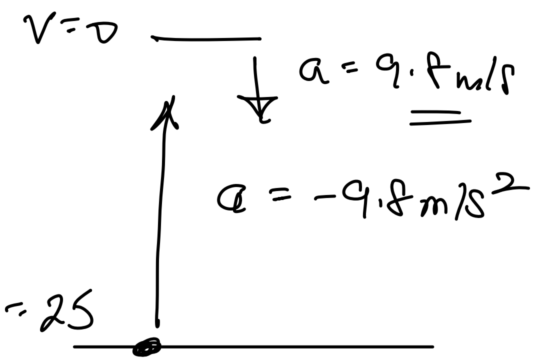
$\therefore x = -\frac{1}{\pi} . \frac{1}{\pi} \cos(\pi t) + 3t + c$

$t = 0, x = 6$ $\quad 6 = -\frac{1}{\pi^2} + c$ $\quad c = 6 + \frac{1}{\pi^2}$

Example – Using antidifferentiation

A cricket ball projected vertically upwards from ground level experiences a gravitational acceleration of 9.8 m/s^2. If the initial speed of the cricket ball is 25 m/s, find:

- its speed after 2 seconds
- its height after 2 seconds
- the greatest height
- the time it takes to return to ground level.



$a = -9.8$

$v = -9.8t + c \qquad t = 0 \quad v = 25$

$25 = c$

$\therefore v = -9.8t + 25 \qquad V_2 = -9.8(2) + 25$

$= 5.4 \text{ m/s}$

$x = -\frac{9.8t^2}{2} + 25t + c$

$x = -4.9t^2 + 25t \qquad x_2 = 30.4 \text{ m}$

$t? \quad v = 0$

$t = \frac{25}{9.8}$

$x = 30.4 \text{ m}$

Learning Objectives: Reviewed

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what each of the following terms mean:
    - Position
    - Displacement
    - Distance
    - Velocity
    - Speed
    - Average velocity
    - Instantaneous velocity
    - Acceleration.
- How to use differentiation and anti-differentiation to find positions, velocities and accelerations.

Questions to complete

The following represents an indication of the minimum number of questions to complete for this exercise. If you choose to do more, then all good. Note that you should also aim to complete some questions from Chapter Reviews too.

Position, velocity and acceleration
Questions: 1, 3, 6, 9, 10, 13, 14, 16, 18, 19