

Inverse functions

Monday, 29 January 2018 8:08 pm

★ By the end of the lesson I would hope that you have an understanding of the concepts below which you can apply to a number of complex questions:

- What an inverse function is
- Difference between an inverse function and inverse relation
- How an inverse function looks on a graph
- Algebraic solution of inverses
- Tips and tricks of inverses.

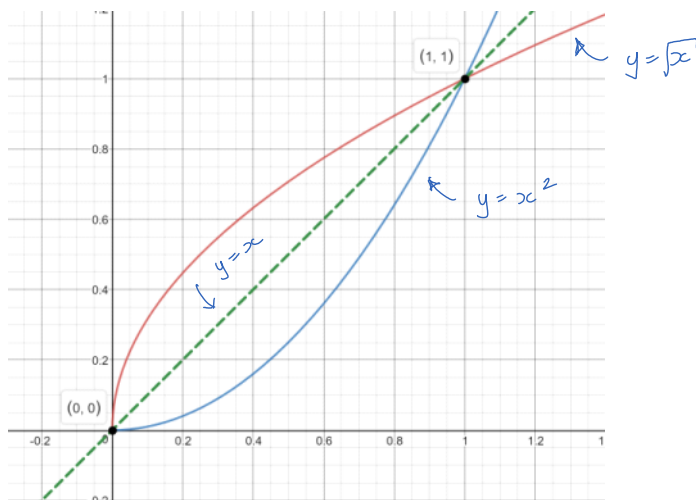
RECAP:

We have been looking at what relations and functions are and how they can be used in Mathematics. We have looked at composite functions and how to add, subtract and multiply functions. The fun really all got too much!

Now we are going to look at one area of Mathematics which tricks people: Inverses!

What is an Inverse:

The simplest way to explain an inverse is that it is when a graph has been reflected in the line $y = x$. That is literally all an inverse is!



Now ... all the if's, but's and maybes ...

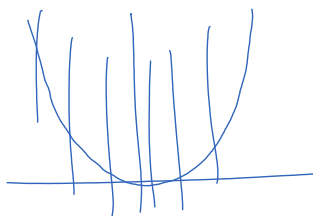
There is a difference between an relation and a function.
We need to be very clear on this to define if a function has an inverse.

This is an example of a relation:

$$x^2 + y^2 = 4$$

This is an example of a function:

$$✓ y = x^2$$



This is an example of a one-to-one function:

$$y = x^2$$
$$y = x^2, x \in \mathbb{R}^+ \cup \{0\}$$

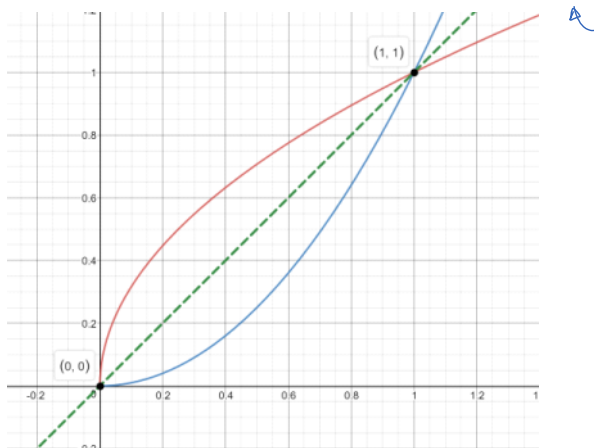


The most important thing to know from this is that only **functions can only have an inverse if they are one-to-one**. All relations can have an inverse.

Confused yet?

Finding the inverse of a relation or function using algebra

Remember that taking the inverse of something is simply reflecting the graph in the line $y = x$.



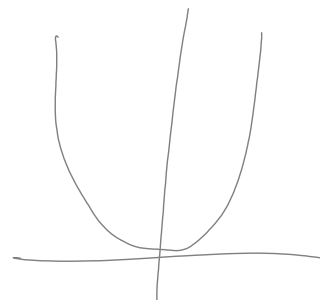
When you reflect in the line $y = x$ you are simply swapping the x and y values.

Hence, this is exactly what you do with an equation where you want to find the inverse of the relation.

Example:

Find the inverse of $y = x^2 - 1$

$$\begin{aligned} x &= y^2 - 1 \\ x + 1 &= y^2 \\ y &= \pm \sqrt{x + 1} \end{aligned}$$



Example:

Find the inverse of $y = 3 \times 4^x - 1$

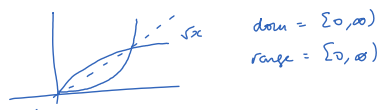
$$\begin{aligned} x &= 3 \times 4^y - 1 \\ x + 1 &= 3 \times 4^y \\ 4^y &= \frac{x + 1}{3} \\ y &= \log_4 \left(\frac{x + 1}{3} \right) \end{aligned}$$

What happens with the domain and range?

You actually can already guess this.

With the original function we can find the domain and range.

If, when we take the inverse, we are swapping a set of x and y values, which just so happens to be the domain and range of a function then it would stand to reason that the **range of the new function is the domain of the old function and vice versa!**

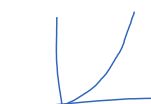
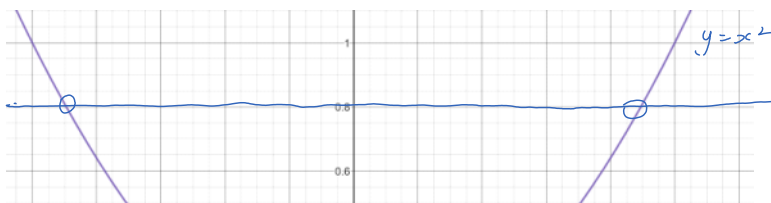


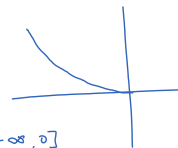
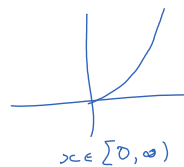
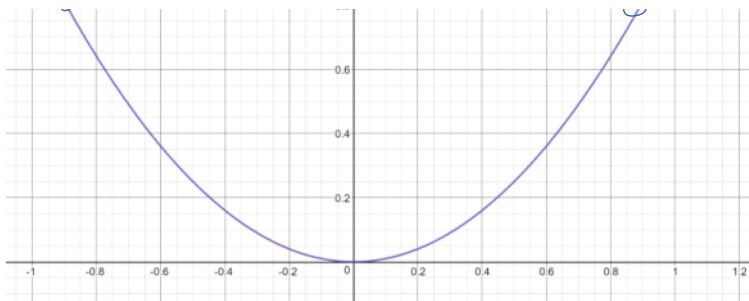
Inverse of a function

To find the inverse of a function it must be one-to-one.

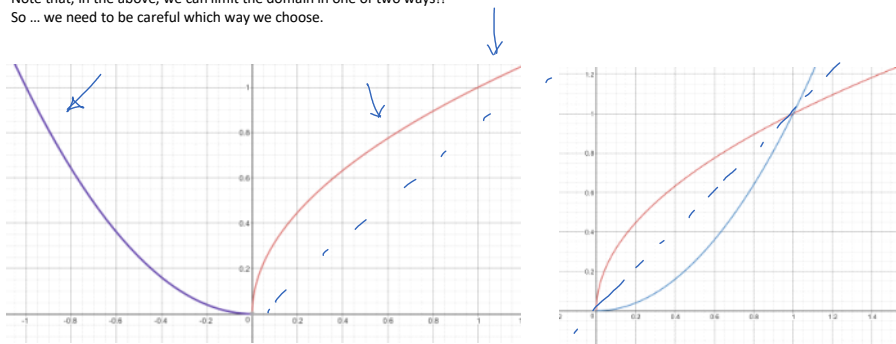
If we look at the function $y = x^2$ we know that the function is not one-to-one.

But ... what can we do to make it one-to-one?





Note that, in the above, we can limit the domain in one of two ways!!
So ... we need to be careful which way we choose.



Notation, notation, notation

We denote the inverse of a function $f(x)$ as $f^{-1}(x)$

This is not to be confused with $(f(x))^{-1}$ which means to take the reciprocal of a function.

$$f^{-1}(x) \quad (f(x))^{-1} \quad \frac{1}{f(x)}$$

Composite functions and inverses

What happens when we start messing about with composite functions?
What if we wanted to do the composite of a function and its inverse?

Example:
If $f(x) = x^2 + 2$ find $f \circ f^{-1}(x)$

$$f(x) = x^2 + 2$$

$$y = x^2 + 2$$

$$x = y^2 + 2$$

$$x - 2 = y^2$$

$$y = \pm \sqrt{x - 2}$$



$$f \circ f^{-1}(x)$$

$$f(x) = x^2 + 2$$

$$f(f^{-1}(x)) = (\cancel{x-2})^2 + 2$$

$$= x - 2 + 2$$

$$= \underline{x}$$

$$f^{-1}(x) \circ f$$

$$f \circ g \quad f(g(x))$$

$$y = \pm \sqrt{x - 2}$$

$$y = \pm \sqrt{(x^2 + 2) - 2}$$

$$y = \pm \sqrt{x^2}$$

$$y = \underline{\underline{x}}$$

Where does an inverse and its original function meet?

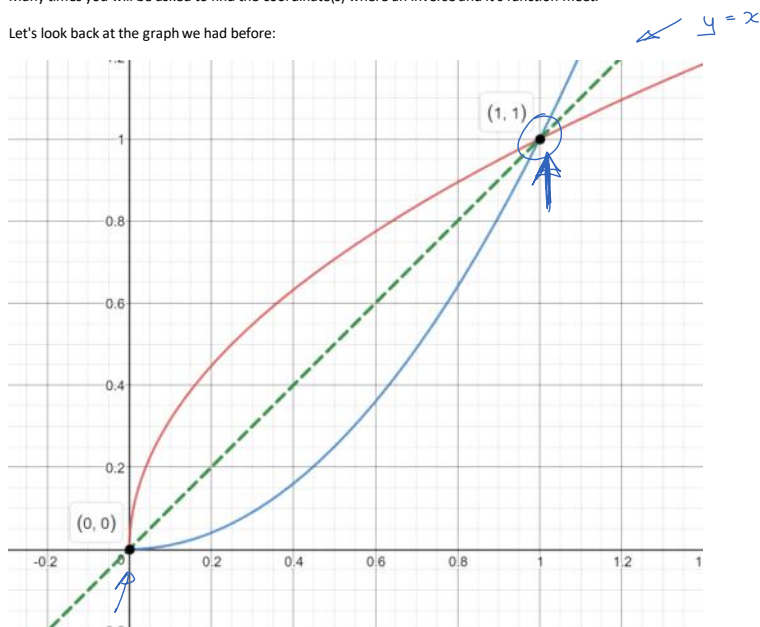
This is a question which appears on exams all the time and so much be a top tip!
Many times you will be asked to find the coordinate(s) where an inverse and its function meet.



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Let's look back at the graph we had before:



$$f(x) = x^2 + 2$$

$$f^{-1}(x) = \pm \sqrt{x-2}$$

$$x^2 + 2 = \pm \sqrt{x-2}$$

$$f(x) = f^{-1}(x) = y$$

$$f(x) = x^2 + 2$$

$$y = x^2 + 2$$

$$x = x^2 + 2$$

$$x^2 - x + 2 = 0$$

Algebra Tricks

You can guarantee that, bearing in mind this stuff seems pretty easy, that they will throw a curve ball in somewhere! And it's going to look something like ...

Example:

Find the inverse of the function shown below and express it in the form: $\frac{a}{x+b} + c$.

Write your answer in function notation

$$y = \frac{x+4}{x+1}$$

$$y = 1 + \frac{3}{x+1}$$

$$x = 1 + \frac{3}{y+1}$$

$$x-1 = \frac{3}{y+1}$$

$$y+1 = \frac{3}{x-1}$$

$$y = \frac{3}{x-1} - 1$$

$$x+1 \overline{) x+4}$$

$$\underline{-x+1}$$

$$3$$

$$f^{-1}: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}, f^{-1}(x) = \frac{3}{x-1} - 1$$

