

Composite functions

★ By the end of the lesson I would hope that you have an understanding of the concepts below which you can apply to a number of complex questions:

- What a composite function is
- How to find $f \circ g$
- How to find $g \circ f$
- How to know if a composite function is defined

RECAP:

We've spent some time looking at all the language we're going to need to navigate the following course.

We've looked at the difference between relations and functions.

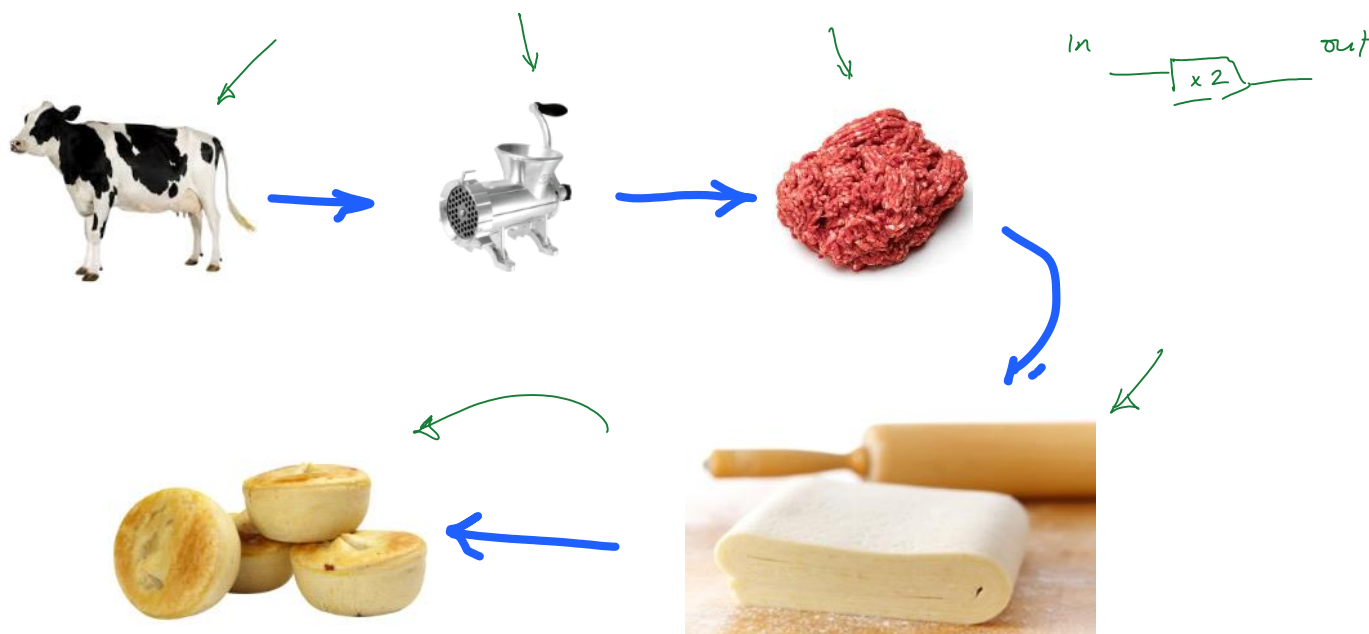
We have looked at what it means to do "addition of ordinates".

Now we move onto the more exciting topic of Composite Functions.

This seems much harder than it really is ... all we need to remember is that what goes in ... must come out ... which goes in ... to come out!

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Imagine I needed to know if the results from one equation could be input into another equation. Perhaps, like a machine, something is put in one end of machine 1, something is done to it, and the resulting product is then put into Machine 2, which does something, and then something else comes out - perhaps the final product!



Hopefully, what comes out of Machine 1 ... will be able to go into Machine 2 ... in it's entirety.

If not ... we would have a problem.

This is the premise for **Composite Functions**.

Functions: Input and Output

Imagine we have the following two functions.

Can we find the domain and range of each one?

What will this help us find?

$$f(x) = 3x + 4$$

$$\text{Domain} = \mathbb{R}$$

$$\text{Range} = \mathbb{R}$$

$$g(x) = x^2$$

$$\text{Domain} = \mathbb{R}$$

$$\text{Range} = \mathbb{R}^+ \cup \{0\}$$

If we want the output of $g(x)$ to be placed into $f(x)$ then we could write it in two ways. The output would then be a whole new function ...

$$f(g(x))$$

$$h(x) = f(g(x))$$

$$h(x) = f \circ g$$

Barry has been at it again. Notice how he's using two things to mean the same!
Dealing with algebra ... we just place the whole of $g(x)$ into $f(x)$

$$\begin{aligned} f(g(x)) &= 3g(x) + 4 \\ &= \underline{3x^2 + 4} \\ &\quad \uparrow \end{aligned}$$

$$\begin{aligned} g \circ f &\quad x^2 \\ g(f(x)) &= (f(x))^2 \\ &= \underline{(3x+4)^2} \\ &\quad \uparrow \end{aligned}$$

Question:

Would $f \circ g$ be the same as $g \circ f$???

How do we know if a composite function is defined/possible?

This is actually pretty easy to determine:

If the output of one function cannot, in its entirety, fit within the domain of the second function, then it is not defined.

Example:

This example has been taken from the Cambridge Essentials Textbook series which I am using to teach this course:

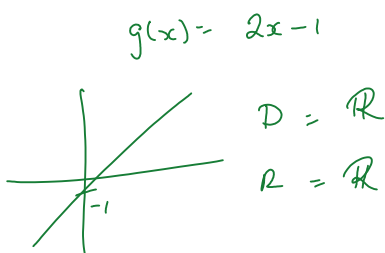
For the functions $g(x) = 2x - 1$, $x \in \mathbb{R}$, and $f(x) = \sqrt{x}$, $x \geq 0$:

- State which of $f \circ g$ and $g \circ f$ is defined.
- For the composite function that is defined, state the domain and rule.



Always a good idea to draw a table!

	Domain	Range
g	$\mathbb{R}$	$\mathbb{R}$
f	$\mathbb{R}^+ \cup \{0\}$	$\mathbb{R}^+ \cup \{0\}$



$$f(x) = \sqrt{x}$$

$$\text{Dom} = \mathbb{R}^+ \cup \{0\}$$

$$\text{Ran} = \mathbb{R}^+ \cup \{0\}$$

$$\text{Dom} = [0, \infty)$$

$$\text{Ran} = [0, \infty)$$

Finding the domain of the composite function ...

$$\begin{aligned} \text{for } f \circ g &\quad \text{Range } g \subseteq \text{dom } f \\ \text{for } g \circ f &\quad \text{Range } f \subseteq \text{dom } g \end{aligned}$$

$$g(x) = 2x - 1$$

$$f(x) = \sqrt{x}$$

$$\begin{aligned} g(f(x)) &= 2f(x) - 1 \\ &= \underline{2\sqrt{x} - 1} \end{aligned} \quad \begin{aligned} \text{dom}(g \circ f) &= \text{dom } f \\ &= \mathbb{R}^+ \cup \{0\} \end{aligned}$$

We note that $\text{dom}(g \circ f) = \text{dom } f$.

Example 2:

For the functions $f(x) = x^2 - 1$, $x \in \mathbb{R}$, and $g(x) = \sqrt{x}$, $x \geq 0$:

- State why $g \circ f$ is not defined.

Dom	Range
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For the functions $f(x) = x^2 - 1, x \in \mathbb{R}$, and $g(x) = \sqrt{x}, x \geq 0$:

- a State why $g \circ f$ is not defined.
b Define a restriction f^* of f such that $g \circ f^*$ is defined, and find $g \circ f^*$.

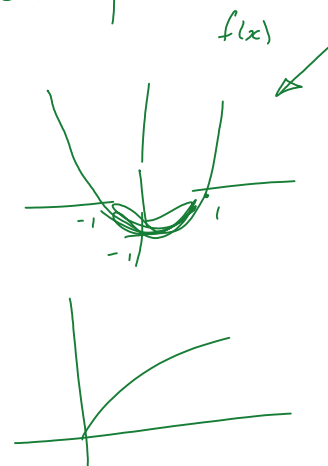
$g \circ f$ Range $f \subseteq \text{domain } g$
 $\Rightarrow [-1, \infty) \not\subseteq [0, \infty)$
 $\therefore g \circ f$ not defined.

$f^* : \mathbb{R} \setminus (-1, 1) \rightarrow \mathbb{R}, f^*(x) = x^2 - 1$

$$g \circ f^* \quad g(f^*(x)) = \sqrt{f^*(x)} \\ = \sqrt{x^2 - 1}$$

$$\text{dom}(g \circ f^*) = \text{dom } f^*(x) \\ = \mathbb{R} \setminus (-1, 1)$$

	Dom	Range
f	$\mathbb{R}$	$[-1, \infty)$
g	$[0, \infty)$	$[0, \infty)$



Important to Note:

Looking at $g \circ f$

If the range of f is **contained within** the domain of g then the composite is defined.
 The domain of the composite function will be the domain of f

Looking at $f \circ g$

If the range of g is **contained within** the domain of f then the composite is defined.
 The domain of the composite function will be the domain of g

In most cases: $f \circ g \neq g \circ f$