Transition matrices and their applications

Thursday, 23 April 2020 7:36 PM

★ By the end of the lesson I would hope that you have an understanding and be able to apply to questions the following concepts:

- Know what a transition matrix is
- Know how to set a transition matrix up
- Know how to interpret a transition matrix
- Know how to use recursion to generate state matrices step-by-step
- Understand what the steady state solution is and how to find it

RECAP

This is the final lesson in this section of the course.
It's the most interesting and potentially the most confusing!
I'm going to build on all the previous work we have done on matrices, so please make sure you've fully understood the work which came before.

Are you ready ...

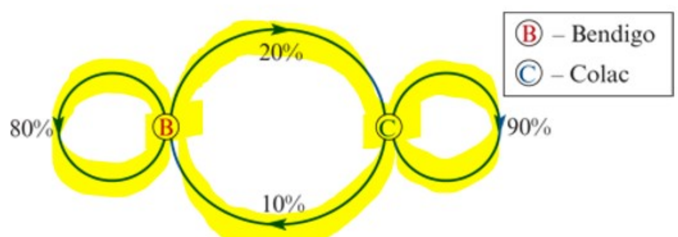
What is a transition matrix?

Basically, it's a way of describing the way in which transitions are made between **two states**.

Imagine that you own two car rental companies.
You need to ensure that you have cars in branch to meet demand.

Here is an example from the Cambridge Further Mathematics Units 3 and 4 textbook.

A car rental firm has two branches: one in Bendigo and one in Colac. Cars are usually rented and returned in the same town. However, a small percentage of cars rented in Bendigo each week are returned in Colac, and vice versa. The diagram below describes what happens on a weekly basis.



What does it mean?

From week to week:

- 0.8 (or 80%) of cars rented each week in Bendigo are returned to Bendigo
- 0.2 (or 20%) of cars rented each week in Bendigo are returned to Colac
- 0.1 (or 10%) of cars rented each week in Colac are returned to Bendigo
- 0.9 (or 90%) of cars rented each week in Colac are returned to Colac.

80% = 0.8

So there are two states:

- The rental car is in Bendigo
- The rental car is in Colac

Rented in

- The rental car is in Bendigo
- The rental car is in Colac

This can be represented as a matrix:

Rented in

$$T = \begin{array}{cc} B & C \\ \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \end{array} \begin{array}{c} B \\ C \end{array} \quad \text{Returning to}$$

Notice the new labels which are added ... "Rented in" and "Returned to"

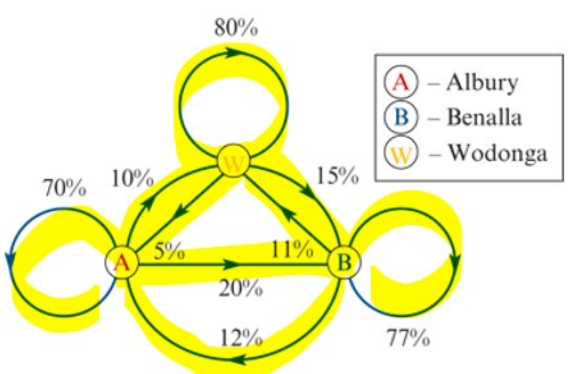
This is going to become important!

Let's practice this a little more.

Examples

The following examples are used, with permission, and have been taken from the Cambridge Further Mathematics Units 3 and 4 textbook.

The diagram gives the weekly return rates of rental cars at three locations: Albury, Wodonga and Benalla. Construct a transition matrix that describes the week-by-week return rates at each of the three locations. Convert the percentages to proportions.



Remember the order of the rows and columns might be important. Rented in

$$T = \begin{array}{ccc} A & B & W \\ \begin{bmatrix} 0.7 & 0.12 & 0.05 \\ 0.2 & 0.77 & 0.15 \\ 0.1 & 0.11 & 0.8 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ W \end{array} \quad \text{Returned.}$$

Remember the order of the rows and columns might be important. Rented

$$T = \begin{array}{c c} \begin{array}{ccc} A & B & W \end{array} & \\ \begin{bmatrix} 0.7 & 0.12 & 0.05 \\ 0.2 & 0.77 & 0.15 \\ 0.1 & 0.11 & 0.8 \end{bmatrix} & \begin{array}{c} A \\ B \\ W \end{array} \end{array}$$

Returned.

What do you notice about the sum of each of the columns?

A factory has a large number of machines. Machines can be in one of two states: operating or broken. Broken machines are repaired and come back into operation, and vice versa. On a given day:

- 85% of machines that are operational stay operating
- 15% of machines that are operating break down
- 5% of machines that are broken are repaired and start operating again
- 95% of machines that are broken stay broken.

Construct a transition matrix to describe this situation. Use the columns to define the situation at the 'Start' of the day and the rows to describe the situation at the 'End' of the day.

[Remember, the order of the columns might be important!]

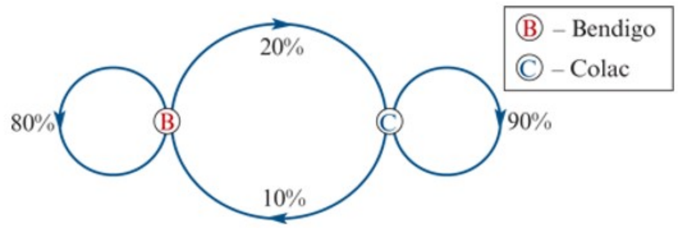
Start

$$T = \begin{array}{c c} \begin{array}{cc} O & B \end{array} & \\ \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} & \begin{array}{c} O \\ B \end{array} \end{array}$$

End.

How do we use the transition matrices in real life?

It's all well and good using percentages, but how many cars are going to be in the car rental business?
What if 50 cars were rented in Bendigo and 40 were rented in Colac?



The transition matrix for the above diagram was:

$$
\begin{array}{ccc}
 & & \textit{Rented in} \\
 & & \begin{array}{cc} \textit{Bendigo} & \textit{Colac} \end{array} \\
\textit{Returned to} & \begin{array}{c} \textit{Bendigo} \\ \textit{Colac} \end{array} & \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}
\end{array}
$$

So, we can find the numbers of cars which will remain in Bendigo, move to Colac, remain in Colac and move to Bendigo.

50 B

40 C

$50 \times 0.8 = 40$

$50 \times 0.2 = 10$

$40 \times 0.9 = 36$

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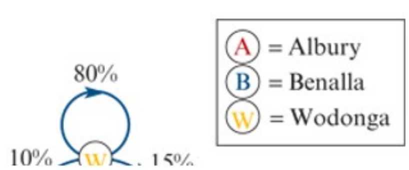
Another example

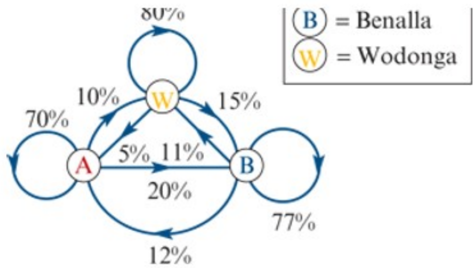
The following examples are used, with permission, and have been taken from the Cambridge Further Mathematics Units 3 and 4 textbook.

The following transition matrix, T, and its transition diagram can be used to describe the weekly pattern of rental car returns in three locations: Albury, Wodonga and Benalla.

$$
T = \begin{array}{cc}
\begin{array}{ccc} A & W & B \end{array} & \\
\begin{bmatrix} 0.7 & 0.05 & 0.12 \\ 0.1 & 0.8 & 0.11 \\ 0.2 & 0.15 & 0.77 \end{bmatrix} & \begin{array}{c} A \\ W \\ B \end{array} \; \textit{Returned to}
\end{array}
$$



$$T = \begin{array}{c} \\ \\ \\ \end{array}\begin{bmatrix} 0.7 & 0.05 & 0.12 \\ 0.1 & 0.8 & 0.11 \\ 0.2 & 0.15 & 0.77 \end{bmatrix}\begin{array}{l} A \\ W \\ B \end{array} \quad \textit{Returned to}$$



0·77

Use the transition matrix T and its transition diagram to answer the following questions.

- What percentage of cars rented in Wodonga each week are predicted to be returned to: 200!
    a. Albury? 5%
    b. Benalla? 15%
    c. Wodonga? 85%
- Two hundred cars were rented in Albury this week. How many of these cars do we expect to be returned to:
    a. Albury? 70% of 200 = 200 × 0·7 = 140
    b. Benalla? 20% of 200 = 40
    c. Wodonga? 20
- What percentage of cars rented in Benalla each week are not expected to be returned to Benalla? 25%
- One hundred and sixty cars were rented in Albury this week. How many of these cars are expected to be returned to either Benalla or Wodonga?

160. 30% of 160 = 48.

I'm planning on being in business for more than one week!

We do indeed plan to be in business for more than one week.
Hence, we need to see what is going to happen to our cars each week.
The numbers we are going to have in each branch is going to change!

Let's see what happens each week ...

To do this we're going to need a recurrence relation.

Remember, from the core data module ...

$$V_0 = Starting\ Value, V_{n+1} = R \times V_n$$

$V_{n+1} = R \times V_n$

Firstly, as we are dealing with Transition Matrices, let's call them T.

Transition matrices are really just **Multipliers**. They are the same as R was in the formula above.

$S_{n+1} = T \times S_n$

The values of V_{n+1} and V_n are also going to be renamed as we call each week a **State**.

$$S_0 = Starting\ state, \qquad S_{n+1} = T \times S_n$$

Let's work out how many vehicles are going to be at the rental locations at the end of the first week.

We know that there are 50 cars in Bendigo and 40 cars in Colac at the start of business.

R

We know that there are 50 cars in Bendigo and 40 cars in Colac at the start of business.

$$S_0 = \begin{bmatrix}50\\40\end{bmatrix} \begin{matrix}B\\C\end{matrix}$$

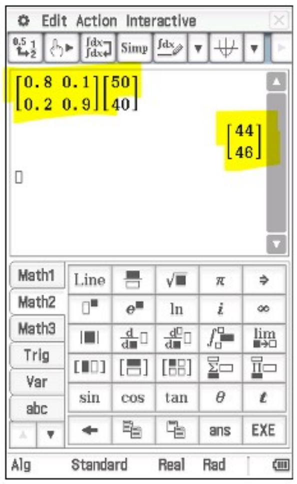
So,

$$S_1 = T \times S_0$$

$S_0 = \begin{bmatrix}50\\40\end{bmatrix}$ $S_{n+1} = T \times S_n$

$S_1 = T \times S_0$

$$S_1 = \begin{bmatrix}0.8 & 0.1\\0.2 & 0.9\end{bmatrix} \times \begin{bmatrix}50\\40\end{bmatrix}$$



B
C

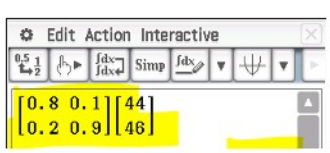
What about at the end of week 2?

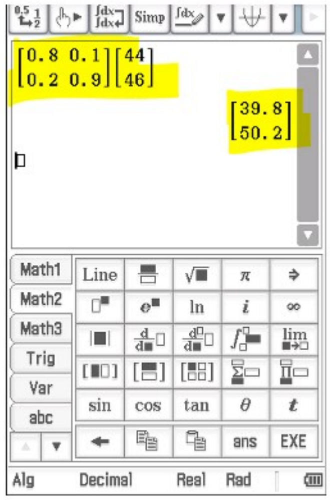
$$S_1 = \begin{bmatrix}44\\46\end{bmatrix}$$

$S_2 = T \times S_1$

$$S_2 = T \times S_1$$

$$S_2 = \begin{bmatrix}0.8 & 0.1\\0.2 & 0.9\end{bmatrix} \times \begin{bmatrix}44\\46\end{bmatrix}$$





What about at the end of week 3?

$$S_2 = \begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$$

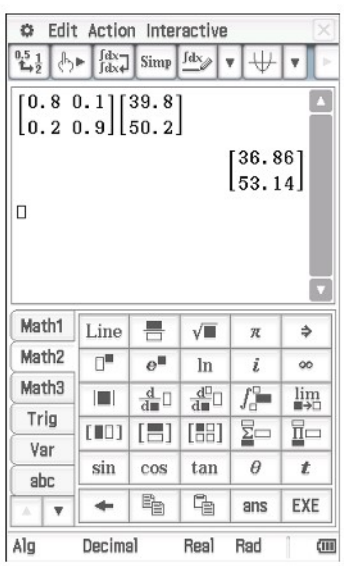
$$S_3 = T \times S_2$$

$$S_3 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \times \begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$$

$S_2 \times T$

$$\begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix} \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

$(2 \times 1)(2 \times 2)$ $\times$



Making things easier

Like we did in the financial section where we turned a recurrence relation into a rule, we can do the same with State Matrices.

We notice that we are doing the same thing over and over again …

$$S_1 = T \times S_0$$

$$S_2 = T \times T \times S_0$$

$S_2 = T \times T \times S_0$

$$S_3 = T \times T \times T \times S_0$$

So, we can make the rule now …

$$S_n = T^n \times S_0$$

$S_n = T^n \times S_0$

Example

The following examples are used, with permission, and have been taken from the Cambridge Further Mathematics Units 3 and 4 textbook.

The factory has a large number of machines. The machines can be in one of two states: operating (O) or broken (B). Broken machines are repaired and come back into operation and vice versa.

Initially, 80 machines are operating and 20 are broken, so:

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \text{ and } T = \begin{array}{cc} & \begin{array}{cc} O & B \end{array} \\ & \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \begin{array}{c} O \\ B \end{array} \end{array}$$

Determine the number of operational and broken machines after 10 days.

$$S_{10} = T^{10} \times S_0$$

$$S_{10} = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}^{10} \begin{bmatrix} 80 \\ 20 \end{bmatrix}$$

$$S_{10} = \begin{bmatrix} 31 \\ 69 \end{bmatrix} \begin{array}{c} O \\ B \end{array}$$

Reaching a "Steady State"

At some point, we are going to reach the "steady state". This is the point where the number of cars in Bendigo and Colac may stay the same after each week.

Let's look at the example using the CAS.

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$$

$$T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

$$S_n = T^n \times S_0$$

Do I have to do this on my calculator pressing all those buttons?

The good news is no!
We need to find a value of n in the formula which is large enough to find the steady state.
Generally we choose a large value of n.

So, let's repeat the above for values of $n = 10, 15, 17 \; and \; 18$

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$$

$$T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

$$S_n = T^n \times S_0$$

S_{10}

S_{18}

S_{15}

S_{17}

What if we want to change things at the end of each week by adding stuff?

Just when you thought it was safe to go back into the water, we change things again (much like we did in the Financial Section of the course).

We now want to start adding (or subtracting) things at the end of each week.

This means we now go back to a recurrence relationship of the form:

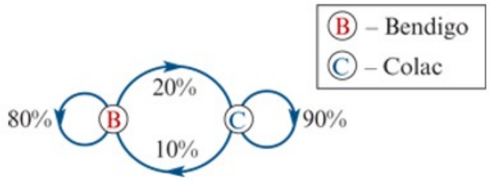
$$S_0 = Start\ Value, \qquad S_{n+1} = T\ S_n + B$$

Where B stands for a matrix where we change the conditions at the end of each week.

Example

The following examples are used, with permission, and have been taken from the Cambridge Further Mathematics Units 3 and 4 textbook.

A rental starts with 90 cars, 50 located at Bendigo and 40 located at Colac.



Cars are usually rented and returned in the same town. However, a small percentage of cars rented in Bendigo are returned in Colac and vice versa. The transition diagram opposite gives these percentages.

To increase the number of cars, two extra cars are added to the rental fleet at each location each week. The recurrence relation that can be used to model this situation is:

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}, \S_{n+1} = TS_n + B \quad \text{where} \quad T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Determine the number of cars at Bendigo and Colac after:

- 1 week
- 2 weeks.

$$S_1 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 50 \\ 40 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 46 \\ 48 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 46 \\ 48 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 44 \\ 54 \end{bmatrix}}}$$