

Matrix arithmetic: the product of two matrices

Thursday, 23 April 2020 7:34 PM

★ By the end of the lesson I would hope that you have an understanding and be able to apply to questions the following concepts:

- How to multiply two matrices together
- Know when you can multiply two matrices together
- Understand what it means for a multiplication to be **defined**
- Understanding that the order of multiplication is really important
- Understand how to use the CAS to multiply two matrices together
- Understand what a summing matrix is

RECAP

We have so far learned what a matrix is and how we can express real world data using matrices.
We have looked at adding and subtracting them and multiplying them by a scalar.
Now ... hold on tight ... this is where it all gets interesting!

The order of matrices now becomes REALLY important

You can only multiply matrices when the number of columns in the first matrix equals the number of rows in the second.

This sounds tough ... but there is an easy way to show if it's going to work.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(2 by 2)

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

(2 by 3)



What about these two?

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(2 by 2)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

(3 by 2)



When two matrices cannot be multiplied by say their **product** is **undefined**.

Remember: Product just means the word multiply.

So, now we know what rules must exist to be able to multiply ... how do we actually multiply them?

Firstly, I want to go back to the order thingy again ...

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 3 \end{bmatrix} = \begin{bmatrix} - & - & - \\ - & - & - \end{bmatrix}$$

(2 by 2) (2 by 3)

$$(2 \text{ by } 2) (2 \text{ by } 3)$$

The outer two numbers give me the dimensions of the matrix I am going to end up with as an answer!
Cool.

So ... let's start easy!

$$\begin{bmatrix} 10 & 4 \\ 9 & 3 \end{bmatrix} \times \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

Check they can be multiplied ...

$$(2 \times 2) (2 \times 1)$$

Now ... we do things in order.

Row 1 times Column 1
Row 2 times Column 1

So, we multiply a row by a column ...

$$\begin{bmatrix} 10 & 4 \\ 9 & 3 \end{bmatrix} \times \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

10
20

Now, for each row and column pairing, we multiply the first two numbers together and the next two numbers together and keep going until we can multiply no more! We then add all the products together.

$$\begin{bmatrix} 10 & 4 \\ 9 & 3 \end{bmatrix} \times \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 58 \\ \end{bmatrix}$$

$$10 \times 5 + 4 \times 2$$

$$50 + 8$$

We then place the answer in the position shown

We then move onto the next row and column pairing. Repeating the process.

$$\begin{bmatrix} 10 & 4 \\ 9 & 3 \end{bmatrix} \times \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 58 \\ 51 \end{bmatrix}$$

$$9 \times 5 + 3 \times 2$$

$$45 + 6$$

We then place the answer in the position shown (and have our finished answer!)

Another example

Let's make it a little more complicated.
Let's look at multiplying the following together

$$\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

11 12

21 22

Check that they can be multiplied and their product defined.

$$(2 \times 2)(2 \times 2)$$

Now, we deal with rows and columns again, but there are more rows and columns.
So, we need to follow an order ..

Row 1, Column 1: Element 11
Row 1, Column 2: Element 12
Row 2, Column 1: Element 21
Row 2, Column 2: Element 22

Row 1, Column 1: Element 11

$$\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 15 & \\ & \end{bmatrix}$$

$$1 \times 3 + 3 \times 4$$

$$3 + 12$$

Row 1, Column 2: Element 12

$$\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 11 \\ & \end{bmatrix}$$

$$1 \times 5 + 3 \times 2$$

$$5 + 6$$

Row 2, Column 1: Element 21

$$\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 11 \\ 27 & \end{bmatrix}$$

$$5 \times 3 + 3 \times 4$$

$$15 + 12$$

Row 2, Column 2: Element 22

$$\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 11 \\ 27 & 31 \end{bmatrix}$$

$$5 \times 5 + 3 \times 2$$

$$25 + 6 = 31$$

Using the CAS to show the order is important

We can use the CAS to do multiplication of Matrices.
You have to make sure the product is defined though before entering the data!

The first screenshot shows the CAS menu with options like Main, eActivity, Statistics, Spreadsheet, Graph&Table, 3D Graph, Geometry, Picture Plot, Interactive DiffCalc, Conics, and DiffEq-Graph.

The second screenshot shows the CAS window with the input $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & 6 \\ 3 & 7 \end{bmatrix}$ and the result $\begin{bmatrix} 10 & 20 \\ 24 & 46 \end{bmatrix}$.

The third screenshot shows the CAS window with the input $\begin{bmatrix} 4 & 6 \\ 3 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and the result $\begin{bmatrix} 22 & 32 \\ 24 & 34 \end{bmatrix}$.

The fourth screenshot shows the CAS window with the input $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & 6 \\ 3 & 7 \end{bmatrix}$ and the result $\begin{bmatrix} 10 & 20 \\ 24 & 46 \end{bmatrix}$.

$$2 \times 3$$

$$3 \times 2$$

The order of multiplication is really important.
If you get the two matrices the wrong way around the answers are all going to be wrong!

The first screenshot shows the CAS window with the input $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & 6 \\ 3 & 7 \end{bmatrix}$ and the result $\begin{bmatrix} 10 & 20 \\ 24 & 46 \end{bmatrix}$.

The second screenshot shows the CAS window with the input $\begin{bmatrix} 4 & 6 \\ 3 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and the result $\begin{bmatrix} 22 & 32 \\ 24 & 34 \end{bmatrix}$.

Example

The following example has been extracted, with permission, from the Cambridge Further Maths Units 3 and 4 Textbook.

$$A = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Which of the following matrix products are defined?

- a AB
- b BC
- c AC

a) $AB = (2 \times 2)(1 \times 2) \quad \times$

b) $BC = (1 \times 2)(2 \times 1) \quad \checkmark$

c) $AC = (2 \times 2)(2 \times 1) \quad \checkmark$

Example

The following example has been extracted, with permission, from the Cambridge Further Maths Units 3 and 4 Textbook.

$$A = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The following matrix products are defined. What is their order?

- a BA
- b BC
- c AC

a) $BA = (1 \times 2)(2 \times 2) \quad (1 \text{ by } 2)$

b) $BC = (1 \times 2)(2 \times 1) \quad (1 \text{ by } 1)$

c) $AC = (2 \times 2)(2 \times 1) \quad (2 \text{ by } 1)$

Example

The following example has been extracted, with permission, from the Cambridge Further Maths Units 3 and 4 Textbook.

$$\text{Evaluate the matrix product } AB, \text{ where } A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}.$$

$$AB = (1 \times 3)(3 \times 1) \quad (1 \times 1)$$

$$\begin{bmatrix} (1)(3) & (1)(4) & (1)(1) \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = [16]$$

$$1 \times 2 + 3 \times 4 + 2 \times 1$$

$$2 + 12 + 2$$

Example

The following example has been extracted, with permission, from the Cambridge Further Maths Units 3 and 4 Textbook.

Evaluate the matrix product AB , where $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$$

$(2 \times 2)(2 \times 1)$

$$1 \times 2 + 0 \times 3$$

$$2 + 0$$

$$2 \times 2 + 3 \times 3$$

$$4 + 9$$

How do we use this stuff in real life?!

Well ... funny you should ask!
Here is a real world example from Cambridge.

Example

The following example has been extracted, with permission, from the Cambridge Further Maths Units 3 and 4 Textbook.

$E = \begin{bmatrix} 25 \\ 40 \end{bmatrix}$ ~~Walk~~ ~~Run~~ Matrix E gives the energy in kilojoules consumed per minute when walking and running.
 $T = \begin{bmatrix} 20 & 40 \end{bmatrix}$ ~~Walk~~ ~~Run~~ Matrix T gives the times (in minutes) a person spent walking and running in a training session.

Compute the matrix product TE and show that it gives the total energy consumed during the training session.

$$TE = \begin{bmatrix} 20 & 40 \end{bmatrix} \begin{bmatrix} 25 \\ 40 \end{bmatrix} = [2100]$$

$$(1 \times 2)(2 \times 1)$$

$$20 \times 25 + 40 \times 40$$

$$500 + 1600$$

$$2100$$

The summing matrix

Another strange one from Barry!

A row matrix or column matrix where all the elements are one is called the summing matrix. The reason we use it is to help us sum rows and columns of matrices.

Example

The following example has been extracted, with permission, from the Cambridge Further Maths Units 3 and 4 Textbook.

Use matrix multiplication to generate a matrix that:

- a displays row sums of the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 7 \\ 3 & 0 & 1 \end{bmatrix}$
- b displays the column sums of the matrix $\begin{bmatrix} 2 & 5 & -1 & -3 & 4 \\ 0 & 6 & 2 & -2 & 3 \end{bmatrix}$.

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 7 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 15 \\ 4 \end{bmatrix}$$

$$(3 \times 3)(3 \times 1)$$

$$1 \times 1 + 3 \times 1 + 0 \times 1$$

$$1 + 3 + 0$$

$$2 \times 1 + 6 \times 1 + 7 \times 1$$

$$2 + 6 + 7$$

$$\begin{bmatrix} 2 & 5 & -1 & -3 & 4 \\ 0 & 6 & 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad & \quad \end{bmatrix}$$

$$(2 \times 5)(5 \times 1) = (2 \times 5)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & -1 & -3 & 4 \\ 0 & 6 & 2 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 11 & 1 & -5 & 7 \end{bmatrix}$$

Using matrices to solve simultaneous equations

This is one of the most awesome uses of matrices that I can recall! If only we taught this in lower year levels!

The following are a pair of simultaneous equations

$$\begin{array}{l} 4x + 2y = 5 \\ 3x + 2y = 2 \end{array}$$

We can actually write them as a product of two matrices

$$\begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$(2 \times 2)(2 \times 1)$$

Let's check by doing it backwards and see if the matrix multiplication gives me back what I am expecting

$$\begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{array}{l} 4x + 2y = 5 \\ 3x + 2y = 2 \end{array}$$