Applications of the inverse matrix: solving simultaneous linear equations

Thursday, 23 April 2020 7:33 PM

By the end of the lesson I would hope that you have an understanding and be able to apply to questions the following concepts:

- Know how to apply the inverse matrix to solve simultaneous equations

RECAP

This is the next video is the series on Matrices.
We took a look, in the last lesson at how we can find the inverse of a matrix by using the following equations:

$$If\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\ then$$

ad − bc

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We looked at what the determinant was (and how to find it using pencil and paper methods and the CAS).

We now are going to use it to solve simultaneous equations ...

RECAP: What are simultaneous equations?

Remember those horrible things we made you solve in Year 10?
The one's which looked like this:

$$4x + 2y = 5$$
$$3x + 2y = 2$$

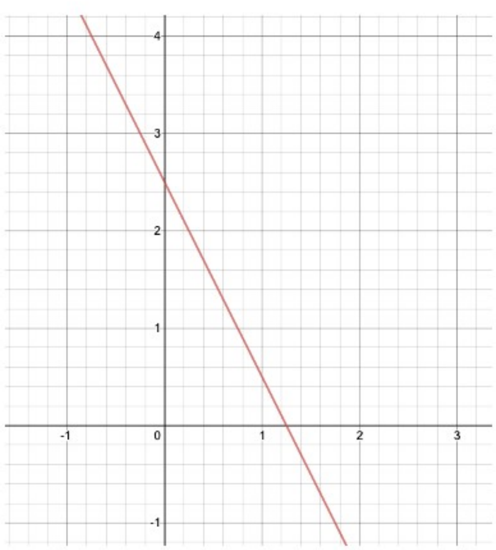
$4x + 2y = 5$

We made you solve them by hand using voodoo!

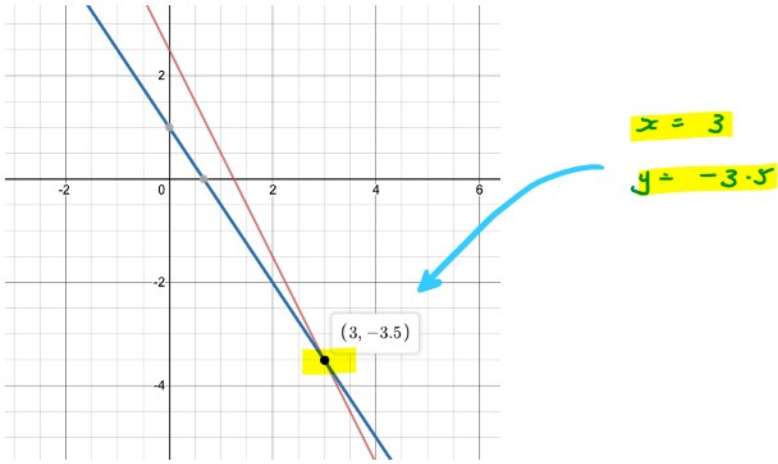
- Substitution
- Elimination

But we might not have told you why we were doing it!

Firstly, the equation $4x + 2y = 5$ is nothing more than an equation of a straight line.

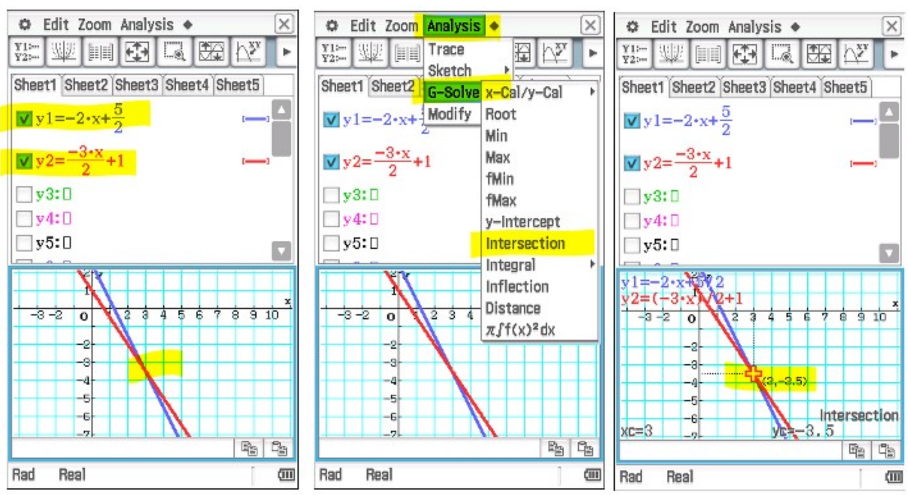


When we place both equations on the same set of axes, we will find (if we are lucky) that they will cross.



The whole reason for solving simultaneous equations is to find the value of x and y which is really just finding the coordinate of where they meet.

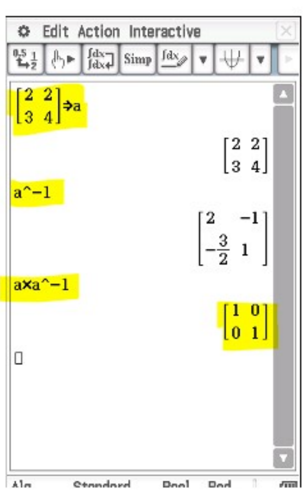
Drawing graphs is much, much easier!
Your CAS can help too …

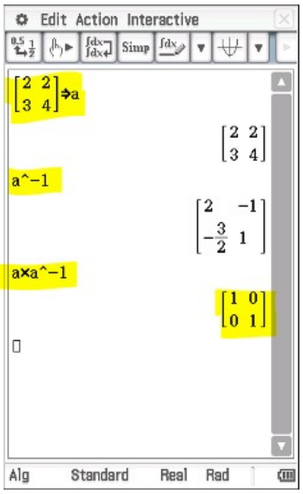


But we're here to solve with matrices

RECAP: Multiplying a matrix by its inverse

Remember, that when we multiply a matrix by its inverse, we always get the identity matrix.
This is now going to be important.





Hence, we know the following $A \times A^{-1} = I$

It can also be shown that $A^{-1} \times A = I$

$$A \times A^{-1} = I$$
$$A^{-1} \times A = I$$

RECAP: Turning simultaneous equations into matrix form

We know, from a previous lesson, that we can turn simultaneous equations into matrix form.

$$4x + 2y = 5$$
$$3x + 2y = 2$$

$$\begin{bmatrix}4 & 2\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}5\\2\end{bmatrix}$$

Let's name each matrix and use some funky (but useful) algebra to show how you can "divide" matrices.

$$\underset{A}{\begin{bmatrix}4 & 2\\3 & 2\end{bmatrix}}\underset{X}{\begin{bmatrix}x\\y\end{bmatrix}} = \underset{C}{\begin{bmatrix}5\\2\end{bmatrix}}$$

$$A.X = C$$
$$A^{-1}.A.X = A^{-1}.C$$
$$I.X = A^{-1}.C$$
$$X = A^{-1}.C$$

So, using the theory on the equations we have:

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}4 & 2\\3 & 2\end{bmatrix}^{-1} \cdot \begin{bmatrix}5\\2\end{bmatrix}$$

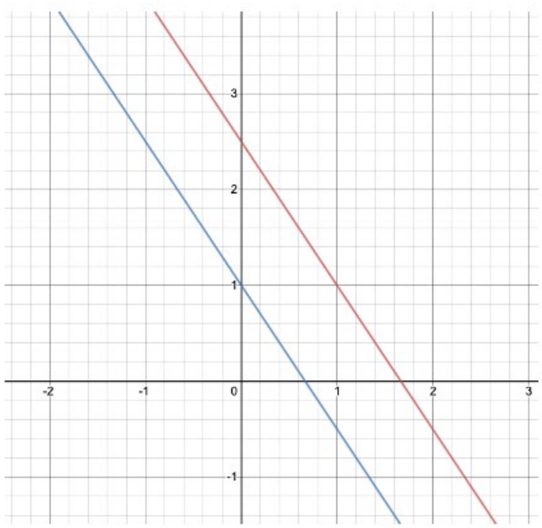
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}^{(-1)} \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -3.5 \end{bmatrix}$$

$x = 3 \quad y = -3.5$

Not all equations have crossing points

What would happen when two lines were parallel?
Would they every have a crossing point?

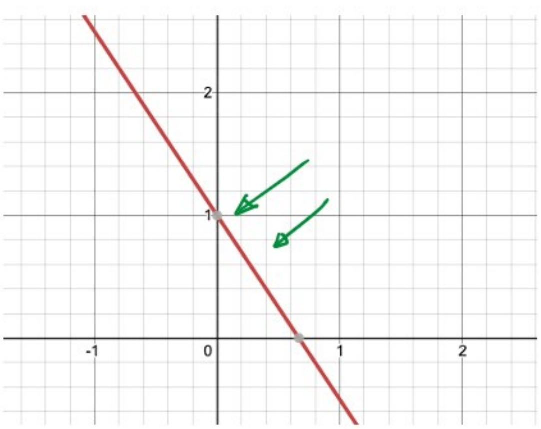


1. $3x + 2y = 5$
2. $3x + 2y = 2$

$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 3 \times 2 - 2 \times 3 \\ &= 6 - 6 \\ &= \underline{\underline{0}} \end{aligned}$$

What if two equations were, in fact, the same line?
How many crossing points would there be?

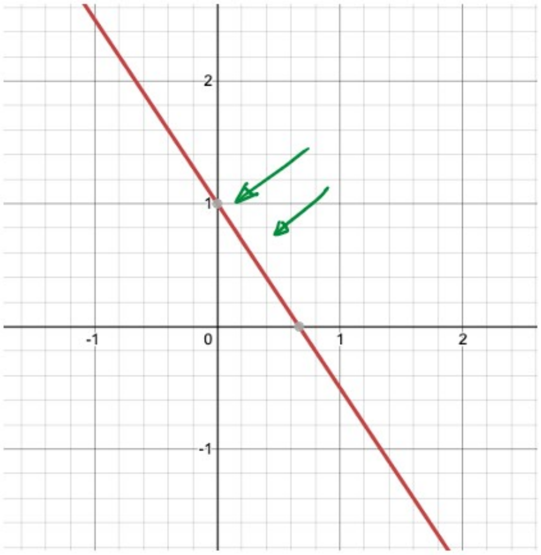


1. $6x + 4y = 4$
2. $3x + 2y = 2$

$\times 2$

$$\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 6 \times 2 - 4 \times 3 \\ &= 12 - 12 \end{aligned}$$



$6x + 4y = 4$

$3x + 2y = 2$ ×2

$$\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\det(A) = 6 \times 2 - 4 \times 3$$
$$= 12 - 12$$
$$= 0$$

Further maths has its own language for the above cases.

Inconsistent: This means the graphs are parallel
Dependent: This means the graphs have lots of crossing points.

How do we know if a graph is inconsistent, dependent or has one solutions?

We know the following graph has no solutions:

$3x + 2y = 5$

$3x + 2y = 2$

What happens if we find the determinant?

$$\det(A) = 3 \times 2 - 2 \times 3$$
$$= 6 - 6$$
$$= 0$$

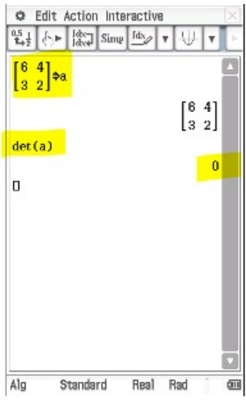
We know the following graphs have lots of solutions, so what happens if we find the determinant?

$6x + 4y = 4$

$3x + 2y = 2$ ×2

$$\det(A) = 6 \times 2 - 4 \times 3$$
$$= 12 - 12$$
$$= 0$$

Remember that we can also use the CAS to find the determinant of matrices!



Examples of solving simultaneous equations

The following examples are used, with permission, and extracted from the Cambridge Further Mathematics Units 3 and 4 textbook.

Solve using matrix methods:

$3x + 4y = 6$
$2x + 3y = 4$

$$\underbrace{\begin{bmatrix}3&4\\2&3\end{bmatrix}}_{A}\begin{bmatrix}x\\y\end{bmatrix}=\underbrace{\begin{bmatrix}6\\4\end{bmatrix}}_{C}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$$

$$x = 2 \quad y = 0$$

Remember, you can use the CAS!

Solve using matrix methods:

$3x + 4.5y = 9$
$2x + 3y = 4$

$$\begin{bmatrix} 3 & 4.5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

A C

$$\det(A) = 3 \times 3 - 2 \times 4.5$$
$$= 9 - 9$$
$$= \underline{\underline{0}}$$

Not all matrices are 2x2!

Solve using matrix methods:

$3x + 4y - 2z = -5$
$2x + 3y = -1$
$x + 2y + 3z = 3$

$$\begin{bmatrix} 3 & 4 & -2 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 7 \\ 0 \end{bmatrix}$$

$$\underline{x = -11 \quad y = 7 \quad z = 0}$$

How is this used in the real world?

I'm glad you asked!
Here is a real world application!

A manufacturer makes two sorts of orange-flavoured chocolates: House Brand and Orange Delights. The number of kilograms of House Brand (x) and the number of kilograms of Orange Delights (y) that can be made from 80 kg of chocolate and 120 kg of orange filling can be found by solving the following pair of equations:

$0.3x + 0.5y = 80$
$0.7x + 0.5y = 120$

Solve for x and y using matrix methods.

$A = \begin{bmatrix} 0.3 & 0.5 \\ \end{bmatrix} \quad C = \begin{bmatrix} 80 \\ 120 \end{bmatrix}$

Solve for x and y using matrix methods.

$$A = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix} \quad C = \begin{bmatrix} 80 \\ 120 \end{bmatrix}$$

$$x = 100 \quad y = 100$$