Analysing reducing-balance loans with recurrence relations

Friday, 10 April 2020 4:40 pm

★ By the end of the lesson I would hope that you have an understanding and be able to apply to questions the following concepts:

- What a reducing balance loan is
- What an amortisation table is
- How to read an amortisation table is
- How to fill in the missing values from an amortisation table

$V_0 = S.R.$

$V_{n+1} = R \times V_n \pm D$

RECAP

We are building on the knowledge gained from Modelling growth and decay using recursion.
We are now in a position to use the information we have gained from previous lessons to look at how to use recursion to model reducing balance loans.

Remember, from a previous lesson

$$V_0 = the\ principal, \qquad V_{n+1} = R \times V_n - D$$

Where:

- V_0 is the principal of the loan or investment
- D is the payment made to clear the loan
- R is the multiplier which is calculated using the following formula

$r =$

$$R = 1 + \frac{r}{100}$$

Where:

- r is the intereset rate per compounding period

Example

The following example has been extracted, with permission, from the Cambridge Further Maths Units 3 and 4 Textbook.

Alyssa borrows $1000 at an interest rate of 15% per annum, compounding monthly.

15% pa

She will repay the loan by making four monthly payments of $257.85.

Construct a recurrence relation to model this loan, in the form

$\frac{15}{12} = 1.25\%$ → r

$$V_0 = the\ principal, \qquad V_{n+1} = RV_n - D$$

where V_n is the balance of the loan after n payments.

$R = 1 + \frac{r}{100}$

$R = 1 + \frac{1.25}{100} = 1.0125$

$V_0 = 1000, \quad V_{n+1} = 1.0125 \times V_n - 257.85$

Now ... we can use the information to analyse a reducing balance load

Now … we can use the information to analyse a reducing balance load

Example

The following example has been extracted, with permission, from the Cambridge Further Maths Units 3 and 4 Textbook.

Alyssa's loan can be modelled by the recurrence relation:

$V_0 = 1000, \qquad V_{n+1} = 1.0125 \times V_n - 257.85$

- Use your calculator to determine recursively the balance of the loan after Alyssa has made each of the four payments.

$V_0 = 1000$

$V_1 = 754.65$

$V_2 = 506.23$

$V_3 = 254.71$

$V_4 = 0.04$

$\underbrace{1000 \times 1.0125}_{\text{add int}} - \underline{257.85}$

- What is the balance of the loan (the amount she still owes) after she has made two payments? Give your answer to the nearest cent.

$506.23

- Is the loan fully paid out after four payments have been made? If not, how much will the last payment have to be to ensure that the loan is fully repaid after four payments?

$257.85 + 0.04 = \underline{\$257.89}$

Amortisation Tables

Loans where you repay them making regular monthly payments until the loan is paid off are called **amortising loans**. You are paying off a small part of the loan each period. Interest is still being added and there is a flow for how you work your way through this process.

The best way of showing it is using an **Amortisation Table**.

The process we are going to follow is:

- Look at the principal (or new balance)
- Make payment
- Add interest
- Reduce principal
- New balance

Let's look at an example.

I've taken out a loan of $1000. The annual interest rate is 15%. I want to make payments of $257.85 per month.
How many months will it be before I clear the loan?

Principal	1000
Interest Rate	0.0125
Term	12
Payment	$257.85

15 % Annual

Payment Number	**Payment**	**Interest**	**Principal Reduction**	**Balance of Loan**
0				1000
1	$257.85	$12.50	$245.35	$754.65
2	$257.85	$9.43	$248.42	$506.23
3	$257.85	$6.33	$251.52	$254.71
4	$257.85	$3.18	$254.67	$0.04
5	$257.85	$0.00	$257.85	-$257.80
6	$257.85	-$3.22	$261.07	-$518.88
7	$257.85	-$6.49	$264.34	-$783.21
8	$257.85	-$9.79	$267.64	-$1,050.85
9	$257.85	-$13.14	$270.99	-$1,321.84
10	$257.85	-$16.52	$274.37	-$1,596.21
11	$257.85	-$19.95	$277.80	-$1,874.01
12	$257.85	-$23.43	$281.28	-$2,155.29

+ int
− pay

$\boxed{257.85} - \boxed{12.50} = \boxed{245.35}$

1000
− 245.35
754.65

257.85
− 9.43
248.42

When will I have paid off my loan?
What will my final payment be?
Why is the interest column having different (and falling) amounts?
Why is the principal reduction growing?

Important Note about the "Cost of the loan"

One of the most important things you need to understand is the cost of the loan.
When you borrow money, you are going to pay back more than you borrowed.
The money you pay on top of the amount you borrowed is the interest.

The total cost of the loan is the **total of all the payments you have made.**

This can also be calculated by adding **the interest over the whole loan** to the **initial amount of the loan**

1000

$\boxed{1463.32}$

Payment	Principal

This can also be calculated by adding **the interest over the whole loan** to the **initial amount of the loan**

Payment Number	Payment	Interest	Principal Reduction	Balance of Loan
0				1000
1	$257.85	$12.50	$245.35	$754.65
2	$257.85	$9.43	$248.42	$506.23
3	$257.85	$6.33	$251.52	$254.71
4	$257.89	$3.18	$254.67	$0.00
Totals	$1,031.44	$31.44		

1031.44
− 1000
31.44

1463.32

Int Cost

1463.32
− 1000
463.32

Example

The following example has been extracted, with permission, from the Cambridge Further Maths Units 3 and 4 Textbook.

A business borrows $10000 at a rate of 8% per annum. The loan is to be repaid by making four quarterly payments of $2626.20. The amortisation table for this loan is shown below.

Payment number	*Payment*	*Interest*	*Principal reduction*	*Balance of loan*
0	0	0.00	0.00	10 000.00
1	2626.20	200	2426.20	7573.80
2	2626.20	151.48	2474.72	5099.08
3	2626.20	101.98	2524.22	2574.86
4	2626.36*	51.50	2574.86	0
	Total		10 000.00	

10504.96
−10 000.00
504.96

*the final payment has been adjusted so that the final balance is zero.

Determine, to the nearest cent:

- the quarterly interest rate

8% p.a ⇒ $\frac{8}{4} = 2\%$

- the interest paid when payment 1 is made using the quarterly interest rate

$ 200

- the principal repaid from payment 2

2626.20 − 151.48 = 2474.72

- the balance of the loan after payment 3 has been made

5099.08 − 2524.22 = 2574.86

- **the total cost of repaying the loan**

$ 504·96

- verify that the total amount of interest paid = total cost of the loan - principal