



Transition matrices: Using recursion

Year 12 General Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson, I would hope that you have an understanding and be able to apply to questions the following concepts:

- To be able to use a matrix recurrence relation: $S_0 = \text{initial state matrix}$, $S_{n+1} = T S_n$, to generate a sequence of state matrices.
- To be able to informally identify the equilibrium state or steady-state matrix in the case of regular state matrices.



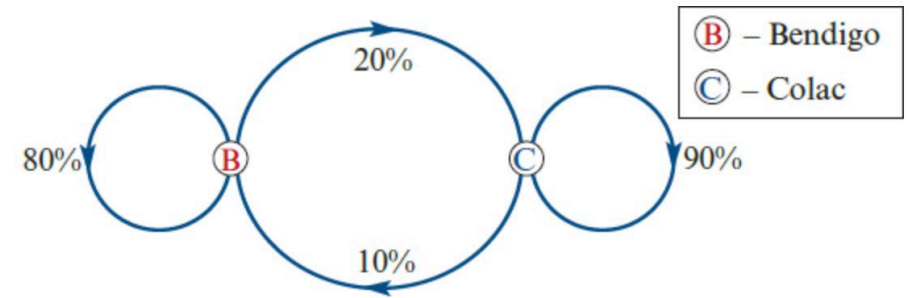
Recap

In the past two lessons we have looked at how we can use transition diagrams and matrices to model real world situations.

Whilst this is all well and good, at the moment we have a little bit of a problem.

The transition matrix only seems to tell us what changes happen in one week. I'm pretty sure I'm going to need to be open much longer than one week for me to make any money!

So, what happens if I want to know how many cars are going to be in Bendigo or Colac after 2, 3, 4, ... weeks?



		Rented in	
		<i>Bendigo</i>	<i>Colac</i>
Returned to	<i>Bendigo</i>	0.8	0.1
	<i>Colac</i>	0.2	0.9



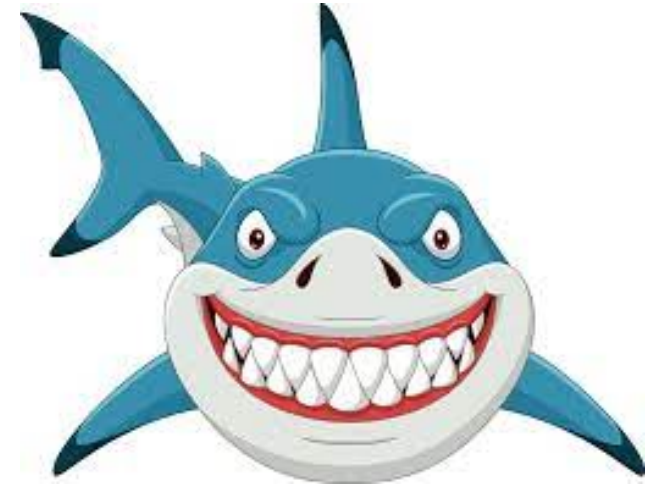
Recurrence relations ... again

Just when you thought it was safe to go into the water again, back come recurrence relationships!!

For financial mathematics we had:

$$V_0 = \text{Principal}, V_{n+1} = R \times V_n \pm D$$

This was simply a way to go from one “week” to another. All we needed to know was our multiplier (R) and if there was some constant amount being added or subtracted.



S_0

$S_0 = \text{Initial state}$

S_n

$S_{n+1} = T.S_n$

S_{n+1}



Let's change some letters!

For Transition Matrices we talk about **STATE**.

The initial STATE is called S_0 .

This might be the number of cars we had when we first opened the business and where they were located.

[Note: The number of cars shouldn't change ... just the locations of them].

The transition matrix is given the letter, T



Opening Day

When I opened by original hire car business in Bendigo, I had enough money to purchase 90 cars.

50 of them were located in Bendigo and the other 40 of them in Colac.

I can represent this as:

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix} \begin{matrix} B \\ C \end{matrix}$$

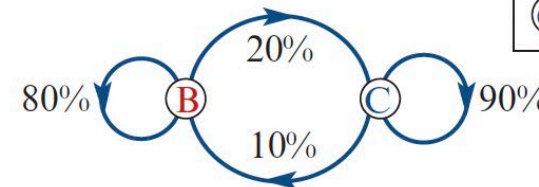
The order of the numbers is really important

We know, from research that the transition matrix, T, is:

$$T = \begin{vmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{vmatrix}$$

Again, the order of the numbers is important

$$T = \begin{matrix} & \begin{matrix} \text{Rented in} \\ B & C \end{matrix} \\ \begin{matrix} B \\ C \end{matrix} \text{ Returned to} & \begin{bmatrix} 0.80 & 0.10 \\ 0.20 & 0.90 \end{bmatrix} \end{matrix}$$



ⓑ – Bendigo
Ⓒ – Colac



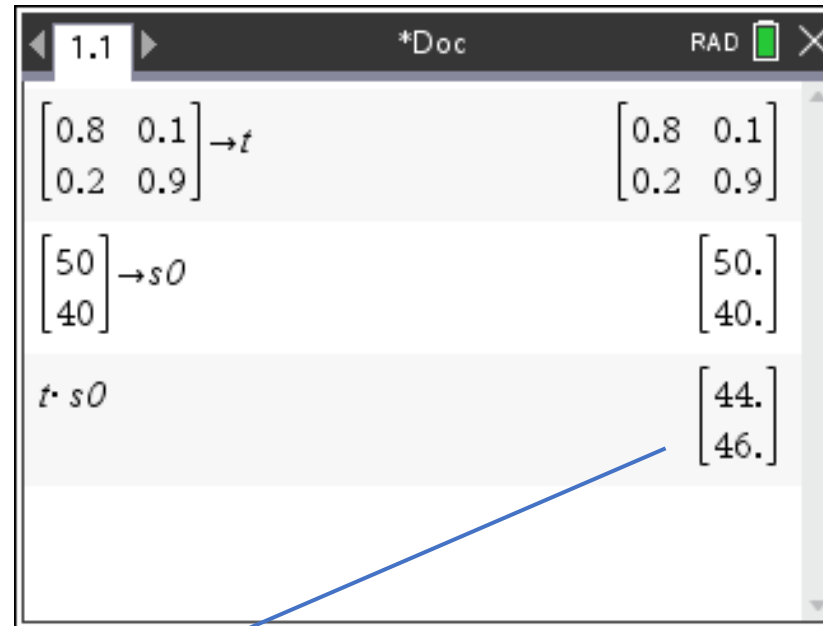
End of the week ... am I rich yet?

I know how many cars were in Bendigo and Colac when I opened the business, but how many are in each location at the end of the week?

This can be found by doing:

$$S_1 = T \times S_0 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \times \begin{bmatrix} 50 \\ 40 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 44 \\ 46 \end{bmatrix} \begin{matrix} B \\ C \end{matrix}$$



Matrix	Operation	Result
$\begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$	$\rightarrow t$	$\begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$
$\begin{bmatrix} 50 \\ 40 \end{bmatrix}$	$\rightarrow s0$	$\begin{bmatrix} 50. \\ 40. \end{bmatrix}$
	$t \cdot s0$	$\begin{bmatrix} 44. \\ 46. \end{bmatrix}$

So, there are going to be 44 cars in Bendigo and 46 cars in Colac.



What about the end of week 2?

To go from one week to the next, we simply take the end of one week and multiply it by the transition matrix

...

So, the end of Week 2 can be found by doing:

$$S_2 = T \times S_1 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \times \begin{bmatrix} 44 \\ 46 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 40 \\ 50 \end{bmatrix} \begin{matrix} B \\ C \end{matrix}$$

1.1	*Doc	RAD	X
[0.2 0.9]		[0.2 0.9]	
$\begin{bmatrix} 50 \\ 40 \end{bmatrix} \rightarrow s0$		$\begin{bmatrix} 50. \\ 40. \end{bmatrix}$	
$t \cdot s0$		$\begin{bmatrix} 44. \\ 46. \end{bmatrix}$	
$t \cdot \begin{bmatrix} 44. \\ 46. \end{bmatrix}$	<i>ans</i>	$\begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$	

We need to be careful with decimals.
There aren't going to be decimal numbers of cars.
So we round, but always make sure the total is 90!

So, 40 cars in Bendigo and 50 cars in Colac now.



And week 3?

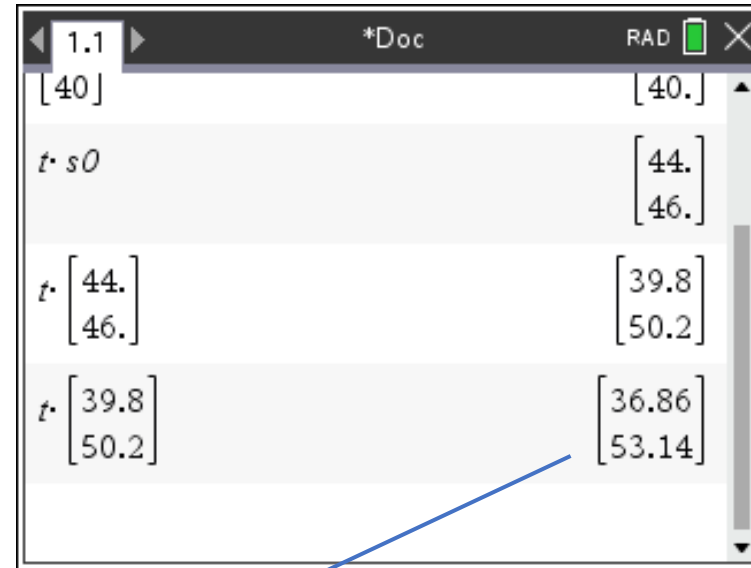
To go from one week to the next, we simply take the end of one week and multiply it by the transition matrix

...

So, the end of Week 3 can be found by doing:

$$S_3 = T \times S_2 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \times \begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$$

Notice we use the decimal values
And not the rounded ones!



The screenshot shows a calculator window titled '*Doc' with the following steps:

1.1	[40.]
t·s0	$\begin{bmatrix} 44. \\ 46. \end{bmatrix}$
t· $\begin{bmatrix} 44. \\ 46. \end{bmatrix}$	$\begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$
t· $\begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$	$\begin{bmatrix} 36.86 \\ 53.14 \end{bmatrix}$

$$S_3 = \begin{bmatrix} 37 \\ 53 \end{bmatrix}$$

We need to be careful with decimals.
There aren't going to be decimal numbers of cars.
So we round, but always make sure the total is 90!

So, 37 cars in Bendigo and 53 cars in Colac now.



Will all the cars end up in Colac then?

What happens if we keep doing this?

Will all the cars end up in Colac? Will I have to close the car business in Bendigo?

We can find the **STEADY STATE** of this system by doing something pretty funky. But, before we do, we need to learn how to make this easier.

We don't want to have to keep doing these types of calculations!!!

$$S_3 = T \times S_2 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \times \begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$$



The rule for finding the state matrix after 'n' steps

This is really building on the same theory we used in Financial Mathematics. So, we should be able to go straight to:

$$S_n = T^n \times S_0$$

To get to any "week" number, we simply raise the transition matrix to the "week number" and multiply it by the **initial state matrix**.

So, if we wanted to find the third week, we can use:

$$S_3 = T^3 \times S_0$$

Same answers!



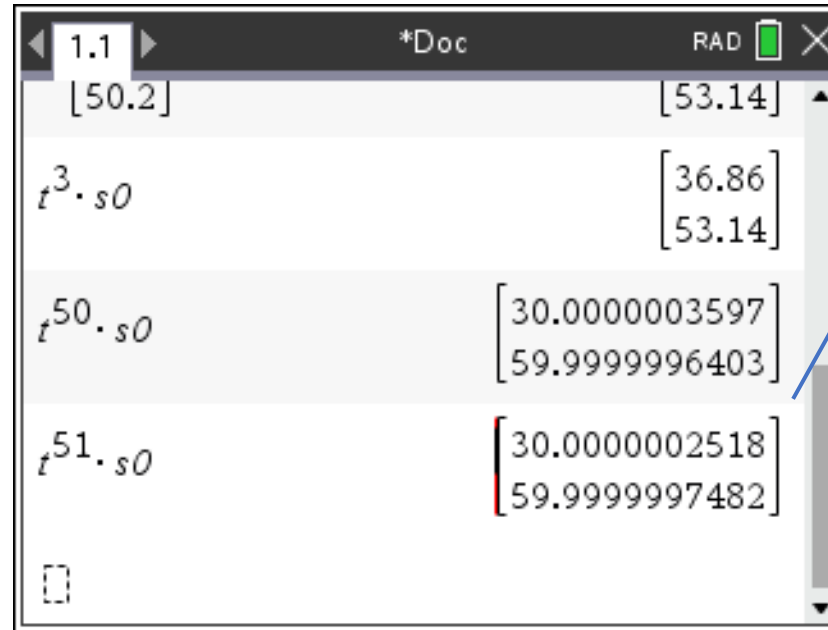
What happens when the value of 'n' gets bigger and bigger?

This is better explained using the CAS so, if you're watching this on the video, then I will demonstrate what happens ..

Otherwise, to find the STEADY STATE of a system, so raise the value of 'n' to something **pretty high**.

Most times you can use 50.

But you must check 51 to make sure there is no significant difference!



n	Matrix
1.1	$\begin{bmatrix} 50.2 & 53.14 \end{bmatrix}$
$t^3 \cdot s0$	$\begin{bmatrix} 36.86 & 53.14 \end{bmatrix}$
$t^{50} \cdot s0$	$\begin{bmatrix} 30.0000003597 & 59.9999996403 \end{bmatrix}$
$t^{51} \cdot s0$	$\begin{bmatrix} 30.0000002518 & 59.9999997482 \end{bmatrix}$

$$S_n = T^n \times S_0$$

We can see there is no significant difference in the whole numbers.

So, we can conclude that, the steady state of the system will always see 30 cars in Bendigo and 60 cars in Colac.

$$\begin{bmatrix} 30 \\ 60 \end{bmatrix} \begin{matrix} B \\ C \end{matrix}$$



Example 1

The factory has a large number of machines. The machines can be in one of two states: operating (O) or broken (B). Broken machines are repaired and come back into operation and vice versa.

At the start, 80 machines are operating and 20 are broken.

Use the recursion relation

$$S_0 = \text{initial value, } S_{n+1} = TS_n$$

where

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

to determine the number of operational and broken machines after 1 day and after 3 days.

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \begin{matrix} O \\ B \end{matrix}$$

$$S_1 = T \cdot S_0 = \begin{bmatrix} 69 \\ 31 \end{bmatrix} \begin{matrix} O \\ B \end{matrix}$$

$$S_3 = T^3 \cdot S_0 = \begin{bmatrix} 53 \\ 47 \end{bmatrix} \begin{matrix} O \\ B \end{matrix}$$



Example 2

The factory has a large number of machines. The machines can be in one of two states: operating (O) or broken (B). Broken machines are repaired and come back into operation and vice versa.

Initially, 80 machines are operating and 20 are broken, so:

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

Determine the number of operational and broken machines after 10 days.

$$S_{10} = T^{10} \times S_0 = \begin{bmatrix} 31 \\ 69 \end{bmatrix}^0 \text{ B}$$



Doing it backwards

Don't you just love Barry?

Not content with trying to confuse you with the whole concept of transition matrices, he now wants to trick you by getting you to go backwards!

So, if you have a transition matrix, T , and S_4 could you reverse engineer the process and go back to S_3 ?

Yup! If you've understood the work from the previous chapter.

$$S_{n-1} = T^{-1} \cdot S_n$$

$$\begin{aligned} S_4 &= T \times S_3 \\ T \times S_3 &= S_4 \\ T^{-1} \times T \times S_3 &= T^{-1} \times S_4 \\ I \times S_3 &= T^{-1} \times S_4 \\ S_3 &= T^{-1} \times S_4 \end{aligned}$$

This is nothing more than some matrix algebra.

You are strongly advised to have this in your summary book somewhere and to understand how to use it!



Example 3

We have a transition matrix $T = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$

and we know that the state matrix $S_4 = \begin{bmatrix} 25 & 587 \\ 34 & 413 \end{bmatrix}$.

Determine S_3 and S_2 .

$$S_3 = T^{-1} \times S_4$$

$$S_3 = \begin{bmatrix} 25 & 290 \\ 34 & 710 \end{bmatrix}$$

$$S_2 = T^{-1} \cdot S_3$$

$$S_2 = \begin{bmatrix} 24 & 300 \\ 35 & 700 \end{bmatrix}$$



Example 4

For the car rental problem:

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Estimate the steady-state solution by calculating S_n for $n = 10, 15, 17$ and 18 .

$$S_{10} = T^{10} \cdot S_0 \\ = \begin{bmatrix} 31 \\ 59 \end{bmatrix}$$

$$S_{15} = \begin{bmatrix} 30 \\ 60 \end{bmatrix}$$

$$S_{18} = \begin{bmatrix} 30 \\ 60 \end{bmatrix}$$

$$S_{17} = \begin{bmatrix} 30 \\ 60 \end{bmatrix}$$



VCAA Questions

A new colony of endangered marsupials is established on a remote island.

For one week, the marsupials can feed from only one of three feeding stations: A , B or C .

On Monday, 50% of the marsupials were observed feeding at station A and 50% were observed feeding at station B . No marsupials were observed feeding at station C .

The marsupials are expected to change their feeding stations each day this week according to the transition matrix T .

$$T = \begin{matrix} & \begin{matrix} \text{this day} \\ A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} \text{ next day} & \begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0.2 & 0.5 & 0.2 \\ 0.4 & 0.4 & 0.6 \end{bmatrix} \end{matrix}$$

$$S_0 = \begin{bmatrix} 50 \\ 50 \\ 0 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix} \quad \leftarrow \text{Monday} \quad S_1 = \begin{bmatrix} 25 \\ 35 \\ 40 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix} \quad \text{Tues}$$

Let S_n represent the state matrix showing the percentage of marsupials observed feeding at each feeding station n days after Monday of this week.

The matrix recurrence rule $S_{n+1} = TS_n$ is used to model this situation.

From Tuesday to Wednesday, the percentage of marsupials who are **not** expected to change their feeding location is

- A. 44.5%
- B. 45%
- C. 50%
- D. 51.5%**
- E. 52%

$$\begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0.2 & 0.5 & 0.2 \\ 0.4 & 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 25 \\ 35 \\ 40 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$$



2021 Further Maths
Paper 1
Question 8
Correct: 29%

VCAA Questions

A market research study of shoppers showed that the buying preferences for the three olive oils, Carmani (C), Linelli (L) and Ohana (O), change from month to month according to the transition matrix T below.

$$T = \begin{array}{ccc} \text{this month} & & \\ C & L & O \\ \left[\begin{array}{ccc} 0.85 & 0.10 & 0.05 \\ 0.05 & 0.80 & 0.05 \\ 0.10 & 0.10 & 0.90 \end{array} \right] & \begin{array}{l} C \\ L \\ O \end{array} & \text{next month} \end{array}$$

$$S_{n+1} = T \cdot S_n$$

The initial state matrix S_0 below shows the number of shoppers who bought each brand of olive oil in July 2021.

$$S_0 = \begin{array}{l} \left[\begin{array}{l} 3200 \\ 2000 \\ 2800 \end{array} \right] \\ C \\ L \\ O \end{array}$$

$$\begin{array}{ccc} C & L & O \\ \left[\begin{array}{ccc} 0.85 & 0.10 & 0.05 \\ 0.05 & 0.80 & 0.05 \\ 0.10 & 0.10 & 0.90 \end{array} \right] & \left[\begin{array}{l} 3200 \\ 2000 \\ 2800 \end{array} \right] & \begin{array}{l} C \\ L \\ O \end{array} \end{array}$$

Let S_n represent the state matrix describing the number of shoppers buying each brand n months after July 2021.

- a. How many of these 8000 shoppers bought a different brand of olive oil in August 2021 from the brand bought in July 2021?

1 mark

$$0.15 \times 3200 + 0.2 \times 2000 + 0.1 \times 2800 = 1160$$



2021 Further Maths
Paper 2
Question 3a
Correct: 28%

VCAA Questions

A market research study of shoppers showed that the buying preferences for the three olive oils, Carmani (C), Linelli (L) and Ohana (O), change from month to month according to the transition matrix T below.

$$T = \begin{array}{c} \begin{array}{ccc} \textit{this month} \\ C & L & O \end{array} \\ \left[\begin{array}{ccc} 0.85 & 0.10 & 0.05 \\ 0.05 & 0.80 & 0.05 \\ 0.10 & 0.10 & 0.90 \end{array} \right] \begin{array}{c} C \\ L \\ O \end{array} \textit{next month} \end{array}$$

The initial state matrix S_0 below shows the number of shoppers who bought each brand of olive oil in July 2021.

$$S_0 = \begin{bmatrix} 3200 \\ 2000 \\ 2800 \end{bmatrix} \begin{array}{c} C \\ L \\ O \end{array}$$

Let S_n represent the state matrix describing the number of shoppers buying each brand n months after July 2021.

- b. Using the rule $S_{n+1} = T \times S_n$, complete the matrix S_1 below.

1 mark

$$S_1 = \begin{bmatrix} 3060 \\ 1900 \\ 3040 \end{bmatrix} \begin{array}{c} C \\ L \\ O \end{array}$$



VCAA Questions

A market research study of shoppers showed that the buying preferences for the three olive oils, Carmani (C), Linelli (L) and Ohana (O), change from month to month according to the transition matrix T below.

$$T = \begin{array}{ccc|c} & \begin{array}{ccc} \text{this month} \\ C & L & O \end{array} & \\ \begin{array}{c} 0.85 \\ 0.05 \\ 0.10 \end{array} & \begin{array}{c} 0.10 \\ 0.80 \\ 0.10 \end{array} & \begin{array}{c} 0.05 \\ 0.05 \\ 0.90 \end{array} & \begin{array}{c} C \\ L \\ O \end{array} \\ \hline & & & \text{next month} \end{array}$$

The initial state matrix S_0 below shows the number of shoppers who bought each brand of olive oil in July 2021.

$$S_0 = \begin{array}{c} 3200 \\ 2000 \\ 2800 \end{array} \begin{array}{c} C \\ L \\ O \end{array}$$

$$\text{Aug} \begin{array}{c} 3060 \\ 1900 \\ 3040 \end{array} \begin{array}{c} C \\ L \\ O \end{array}$$

Let S_n represent the state matrix describing the number of shoppers buying each brand n months after July 2021.

- c. Consider the shoppers who were expected to buy Carmani olive oil in August 2021.

What percentage of these shoppers also bought Carmani olive oil in July 2021?

Round your answer to the nearest percentage.

1 mark

$$\frac{0.85 \times 3200}{3060} = \frac{2720}{3060} \times 100 = \underline{\underline{89\%}}$$



VCAA Questions

Consider the matrix recurrence relation below.

$$S_0 = \begin{bmatrix} 30 \\ 20 \\ 40 \end{bmatrix}, \quad S_{n+1} = T S_n \quad \text{where } T = \begin{bmatrix} j & 0.3 & l \\ 0.2 & m & 0.3 \\ 0.4 & 0.2 & n \end{bmatrix}$$

$$\begin{aligned} j &= 0.4 \\ m &= 0.5 \\ l &= 0.6 \\ n &= 0.1 \end{aligned}$$

Matrix T is a regular transition matrix.

Given the information above and that $S_1 = \begin{bmatrix} 42 \\ 28 \\ 20 \end{bmatrix}$, which one of the following is true?

- A. $m > l$
- B. $j + l = 0.7$
- C. $j = n$
- D. $j > m$
- E. $l = m + n$

$$\begin{bmatrix} 0.4 & 0.3 & l \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.2 & n \end{bmatrix} \begin{bmatrix} 30 \\ 20 \\ 40 \end{bmatrix} = \begin{bmatrix} 42 \\ 28 \\ 20 \end{bmatrix}$$

$$0.4 \times 30 + 0.3 \times 20 + l \times 40 = 42$$

$$l \times 40 = 42 - 18$$

$$l \times 40 = 24$$

$$l = 0.6$$



2020 Further Maths
Paper 1
Question 10
Correct: 41%

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