

The Binomial Distribution

Year 11
Mathematical Methods

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Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Year 11 Mathematical Methods course.

- Understand what it means to conduct a binomial experiment
- Understand the ideas of successes and failures
- Understand what it means to be a trial
- Understand what it means to be a Binomial Distribution
- Know how to apply the distribution to a range of probability questions



RECAP

This is the final part of this section of the course where we look at the idea of binomial distributions and experiments in which there can only be two outcomes.

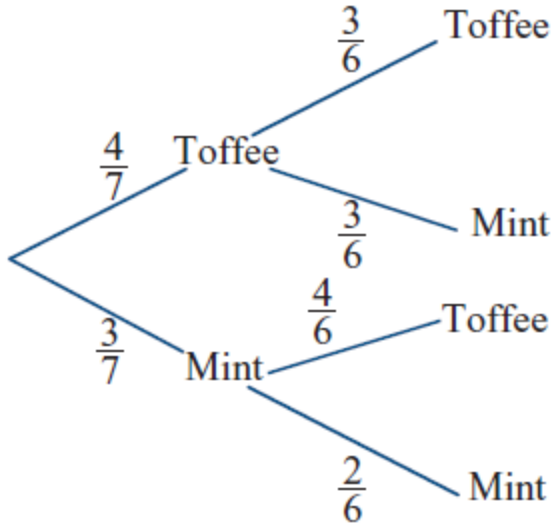


With and without replacement

In the previous lesson we learned that we need to be careful about conducting experiments where the tree diagram is created with and without replacement of items. This was really a recap of the work from Years 9 and 10.

We are, for this section, going to look at experiments where we will be sampling with replacement.

A jar containing three mints and four toffees. Suppose Bob selects two lollies (without looking) and that the random variable X is the number of mints he selects, which can take the value 0, 1 or 2.



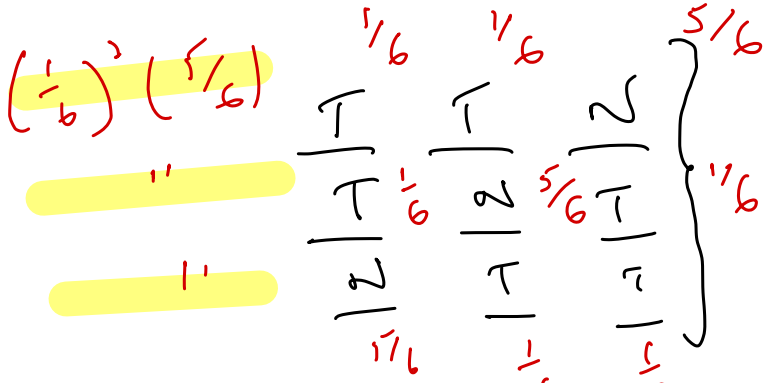
x	0	1	2
$\Pr(X = x)$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$





An easier way to find the individual probabilities

Suppose a fair six-sided die is rolled three times and a random variable X is defined as the number of 3s observed. Let T represent a 3, and let N represent not a 3. Consider all the possible outcomes and the probability for each outcome.



Outcome	Number of 3s	Probability
TTT	$X = 3$	$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$
TTN	$X = 2$	$\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}$
TNT	$X = 2$	$\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}$
NTT	$X = 2$	$\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}$
TNN	$X = 1$	$\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}$
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NNT	$X = 1$	$\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$
NNN	$X = 0$	$\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$

$$\Pr(X = 3) = \left(\tfrac{1}{6}\right)^3 \qquad {}^3C_3$$

$$\Pr(X = 2) = 3 \times \left(\tfrac{1}{6}\right)^2 \times \tfrac{5}{6} \qquad {}^3C_2$$

$$\Pr(X = 1) = 3 \times \tfrac{1}{6} \times \left(\tfrac{5}{6}\right)^2 \qquad {}^3C_1$$

$$\Pr(X = 0) = \left(\tfrac{5}{6}\right)^3 \qquad {}^3C_0$$

x	0	1	2	3
$\Pr(X = x)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$



A brief explanation of Combinatorics

Many students don't quite ever get to grips with what the following means:

$$^{n}C_{x}$$

It means, with 'n' individual items, how many ways are there of me choosing 'x' from it.

Example:

$$^{3}C_{2}$$

Means if I have three items, how many ways are there of choosing 2 items.

$$^{3}C_{2} = 3$$



An easy formula

We can use the following formula to help us find the individual probabilities for a binomial distribution so long as we know some important information:

p = The probability of a successful outcome in a single trial
n = The number of identical trials which were conducted
x = The individual trial you are looking at a result for

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\Pr(X = x) = {}^{n}c_x \, p^x \, (1-p)^{n-x}$$

$${}^{n}C_{x} \cdot p^{x}$$

$${}^{3}C_{2} \left(\frac{1}{6}\right)^{2} \left(\frac{5}{6}\right)^{1}$$



Example

Rainfall records for the city of Melbourne indicate that, on average, the probability of rain falling on any one day in November is 0.4. Assuming that the occurrence of rain on any day is independent of whether or not rain falls on any other day, find the probability that rain will fall on any three days of a chosen week.

$p = 0.4$

$\bar{p} = 0.6$

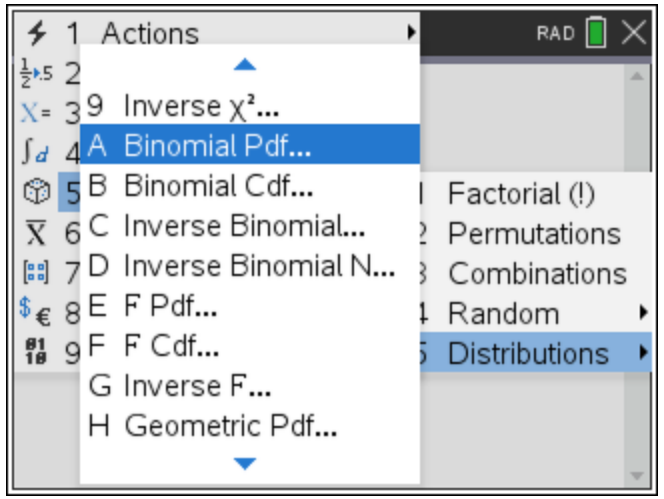
$${}^{7}C_{3} \cdot (0.4)^{3}(0.6)^{4} = 0.2903$$

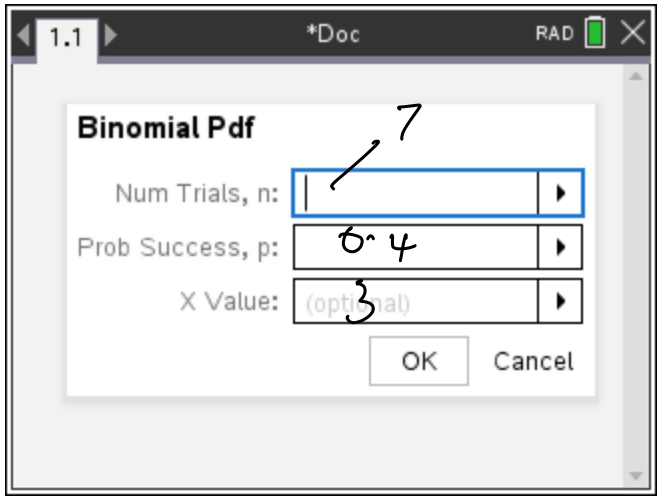
$$\underline{R}\ \underline{R}\ \underline{R}\ \underline{\bar{R}}\ \underline{\bar{R}}\ \underline{\bar{R}}\ \underline{\bar{R}}$$



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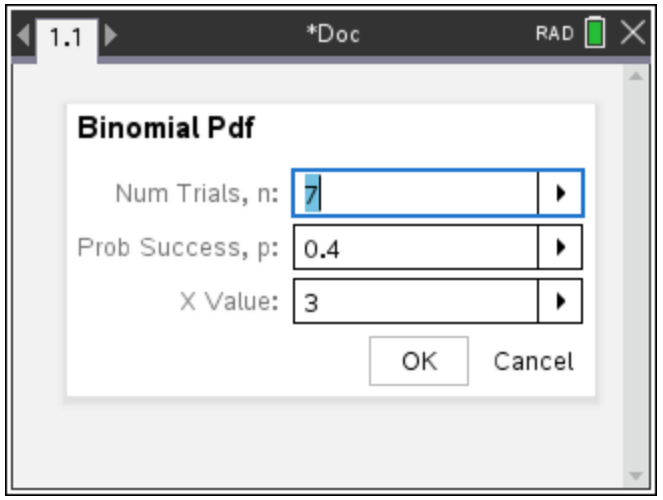


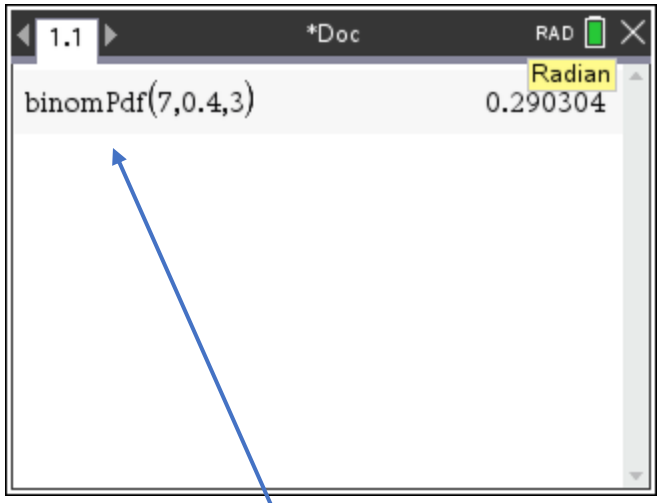




Example

Rainfall records for the city of Melbourne indicate that, on average, the probability of rain falling on any one day in November is 0.4. Assuming that the occurrence of rain on any day is independent of whether or not rain falls on any other day, find the probability that rain will fall on any three days of a chosen week.





It is actually easier to copy, paste and change the numbers for subsequent calculations than keep using the Menu function!



Example

For the situation described in the previous example, use a CAS calculator to find the probability that:

- rain will fall on any three days of a chosen week
- rain will fall on no more than three days of a chosen week
- rain will fall on at least three days of a chosen week.



Example

For the situation described in the previous example, use a CAS calculator to find the probability that:

- rain will fall on any three days of a chosen week

$$\Pr(X = 3)$$

Binom CDF

- rain will fall on no more than three days of a chosen week

$$\Pr(X \leq 3)$$

- rain will fall on at least three days of a chosen week.

$$\Pr(X \geq 3)$$



Example

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- rain will fall on any three days of a chosen week

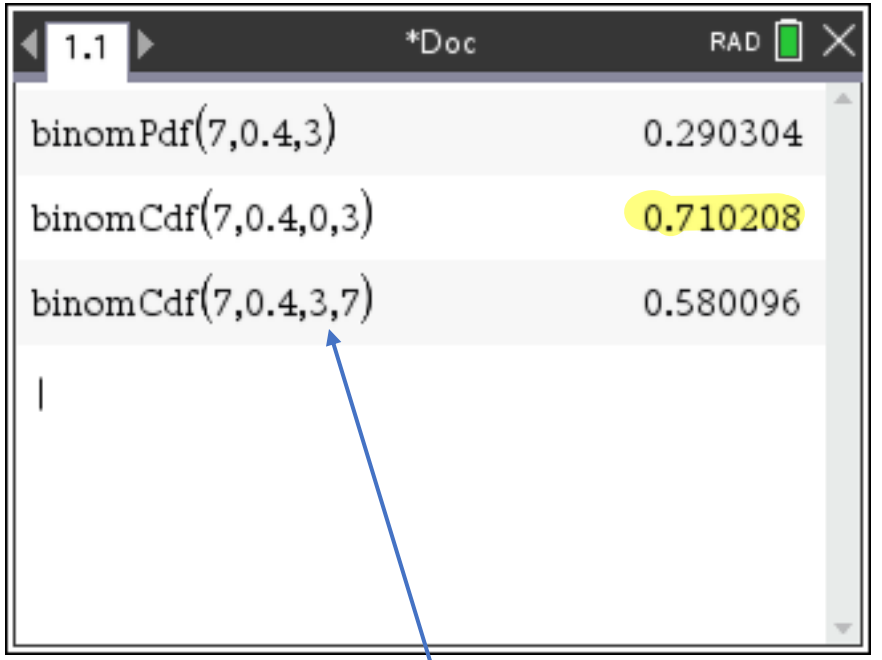
$$\Pr(X = 3)$$

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$$\Pr(X \leq 3)$$

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$$\Pr(X \geq 3)$$

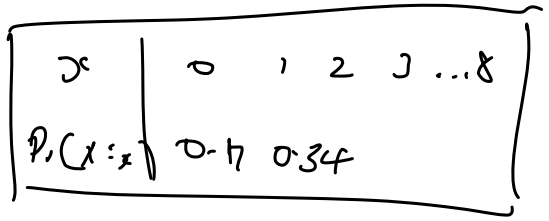


Notice for **binomCDF** that there is an extra argument. This is the lower and upper bounds in which you wish to add all the individual probabilities.



Seeing what a distribution looks like

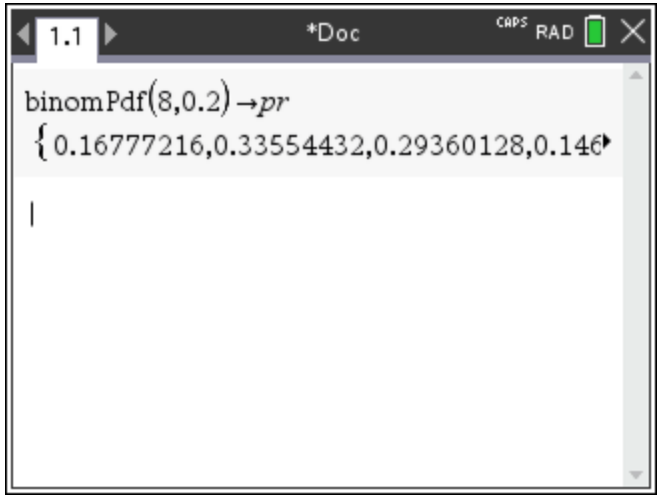
This is a pretty interesting exercise to be able to use the CAS to visualise what the distribution looks like. Remember, these are **discrete** outcomes and so there will be no connector between the points.



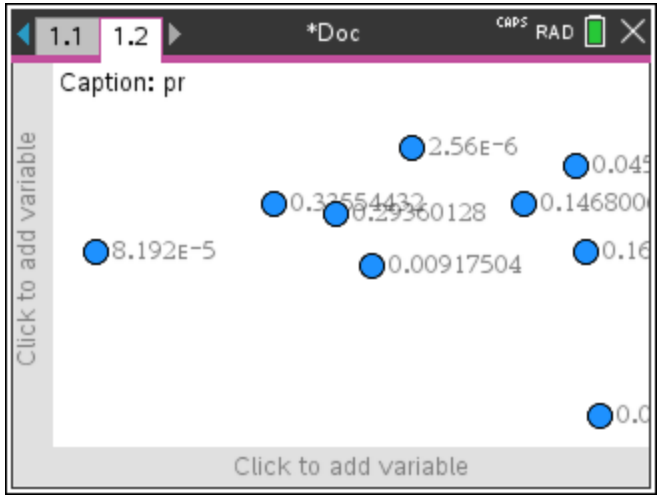
Use a CAS calculator to plot the following probability distribution function:

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots, n$$

for $n = 8$ and $p = 0.2$.



Note: When you don't give a value for 'x' with **binomPdf**, it gives all values! Very cool.



To be able to plot the probabilities we need to have two numerical variables along the axes. We have the probabilities (which are always on the RV).



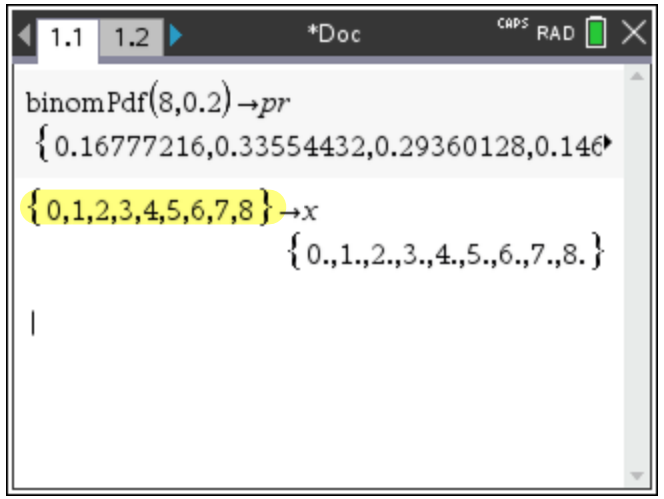
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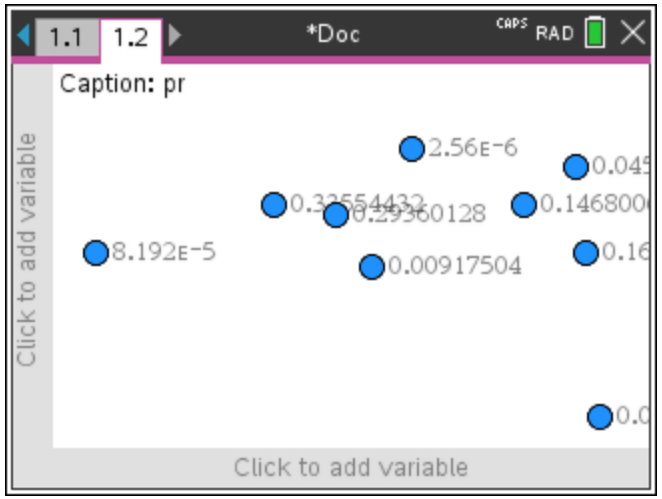
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You need to tell the CAS what to plot where!



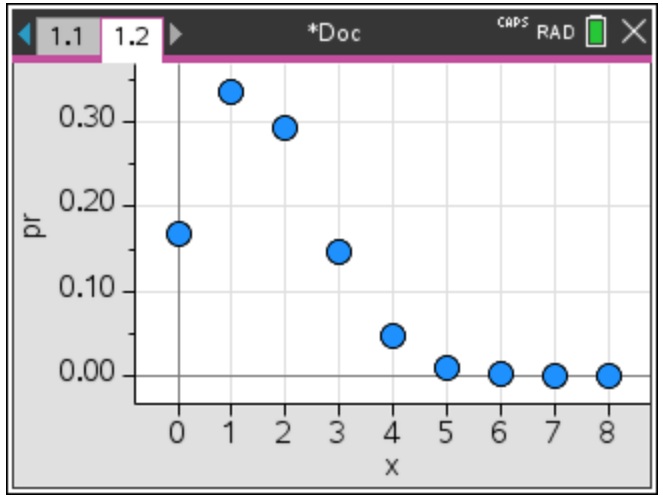
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This is a pretty interesting exercise to be able to use the CAS to visualise what the distribution looks like. Remember, these are **discrete** outcomes and so there will be no connector between the points.

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for $n = 8$ and $p = 0.2$.





Example

The probability of winning a prize in a game of chance is 0.25. What is the least number of games that must be played to ensure that the probability of winning at least twice is more than 0.9?

$p = 0.25$

$n = ?$

$Pr > 0.9$

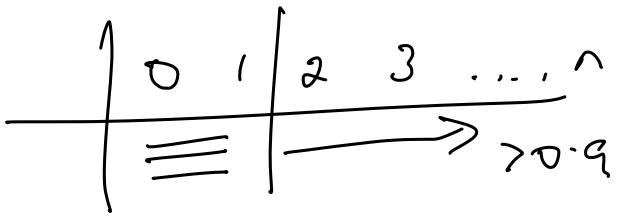
We need to think of this question in a different way as we don't know the number of games (n).

It wants us to find 'n' such that the $\Pr(X \geq 2) > 0.9$.
How would we do this?
Can we turn the question on its head?

$Pr(X \geq 2) > 0.9$

$Pr(X < 2) < 0.1$

$Pr(X = 0) + Pr(X = 1)$





Example

The probability of winning a prize in a game of chance is 0.25. What is the least number of games that must be played to ensure that the probability of winning at least twice is more than 0.9?

We can probably find the following much easier using the following and some basic ideas of combinations:

$\Pr(X < 2) < 0.1$



Example

The probability of winning a prize in a game of chance is 0.25. What is the least number of games that must be played to ensure that the probability of winning at least twice is more than 0.9?

We can probably find the following much easier:

$$\Pr(X < 2) = \Pr(X = 0) + \Pr(X = 1)$$

$$\Pr(X = 0) = \binom{n}{0} 0.25^0 (0.75)^n$$

$$\Pr(X = 1) = \binom{n}{1} 0.25^1 (0.75)^{n-1}$$



Example

The probability of winning a prize in a game of chance is 0.25. What is the least number of games that must be played to ensure that the probability of winning at least twice is more than 0.9?

We can probably find the following much easier:

$$\Pr(X < 2) = \Pr(X = 0) + \Pr(X = 1)$$

$$\binom{n}{0} 0.25^0 (0.75)^n + \binom{n}{1} 0.25^1 (0.75)^{n-1} < 0.1$$

$$(0.75)^n + n \times 0.25^1 (0.75)^{n-1} < 0.1$$



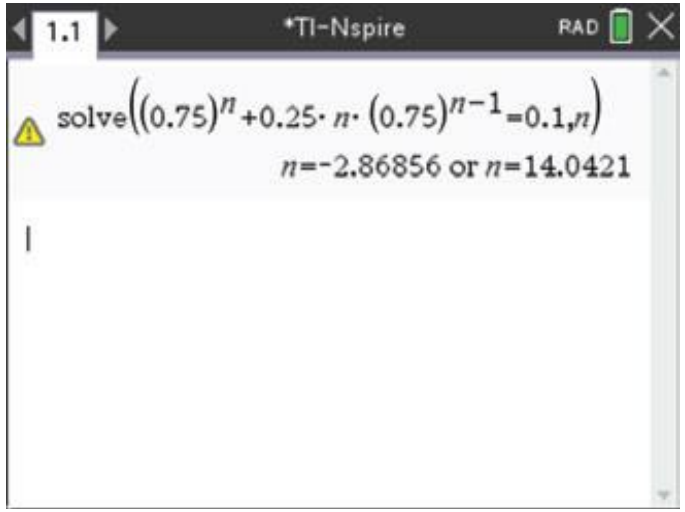
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We can probably find the following much easier:

$$\Pr(X < 2) = \Pr(X = 0) + \Pr(X = 1)$$

$$(0.75)^n + n \times 0.25^1 (0.75)^{n-1} < 0.1$$



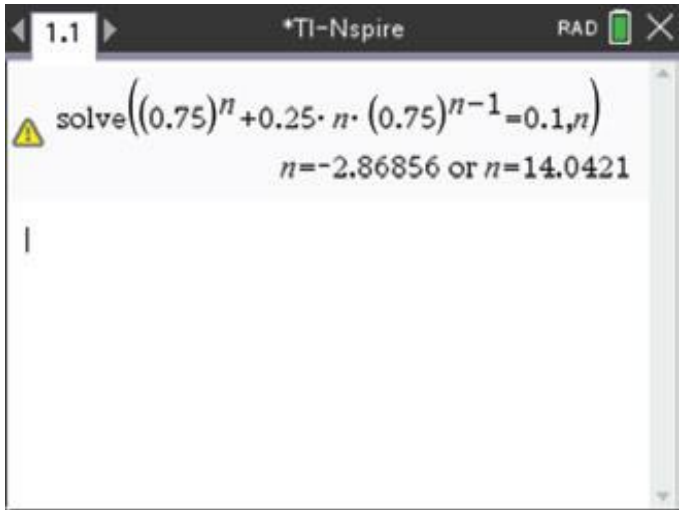


Example

The probability of winning a prize in a game of chance is 0.25. What is the least number of games that must be played to ensure that the probability of winning at least twice is more than 0.9?

We can probably find the following much easier:

$$\Pr(X < 2) = \Pr(X = 0) + \Pr(X = 1)$$



Interpreting the result is then really, really important! We can't play a decimal number of games and so we need to work out if we round up OR down.

Will the answer be 14 or 15?



Example

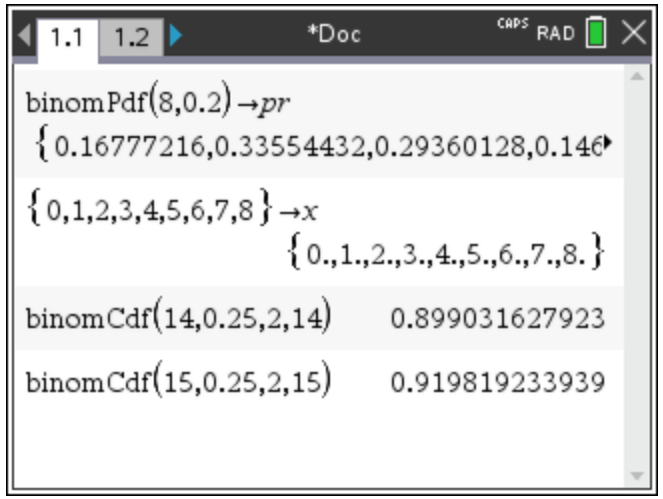
The probability of winning a prize in a game of chance is 0.25. What is the least number of games that must be played to ensure that the probability of winning at least twice is more than 0.9?

$\Pr(X \geq 2) > 0.9$

$\Pr(X < 2) < 0.1$

We can probably find the following much easier:

$$\Pr(X < 2) = \Pr(X = 0) + \Pr(X = 1)$$



If all else fails, you can use the CAS to check!



Example 2

The probability of an archer obtaining a maximum score from a shot is 0.4. Find the probability that out of five shots the archer obtains the maximum score:

a three times

b three times, given that she obtains the maximum score at least once.

$p = 0.4$

$n = 5$

a. $P(X = 3) = {}^5C_3 \cdot (0.4)^3 (0.6)^2 = 0.2304$

b. $P(X = 3 \mid X \geq 1) = \dfrac{Pr(X = 3 \cap X \geq 1)}{Pr(X \geq 1)} = \dfrac{Pr(X = 3)}{Pr(X \geq 1)} = \dfrac{0.2304}{0.92224}$

$= 0.2498$



Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Year 11 Mathematical Methods course.

- Understand what it means to conduct a binomial experiment
- Understand the ideas of successes and failures
- Understand what it means to be a trial
- Understand what it means to be a Binomial Distribution
- Know how to apply the distribution to a range of probability questions



Questions to complete

The following are the minimum number of questions you are expected to answer. There is nothing wrong with answering more!

Ex 11C
Questions: ALL





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