Differential equations involving a function of the independent variable

Year 12 Specialist Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Know how to solve differential equations of the form $\frac{dy}{dx} = f(x)$
- Understand that a general solution will relate to a family of curves.
- Know that a particular solution will find the solution as one curve.
- Know how to solve differential equations of the form $\frac{d^2y}{dx^2} = f(x)$

Recap of past learning

In the past lesson we looked at proving that a particular solution fits a particular differential equation. We now seem to go backwards in our learning as we return to the very basics of integration.

We know that we can solve functions of the form:

$$\frac{dy}{dx} = f(x)$$

By:

$$\int \frac{dy}{dx}\,dx = \int f(x)\,dx$$

Hence

$$y = \int f(x)\,dx$$

Starting with something simple

Find the general solution of each of the following:

- $\frac{dy}{dx} = x^4 - 3x^2 + 2$
- $\frac{dy}{dt} = sin(2t)$
- $\frac{dx}{dt} = e^{-3t} + \frac{1}{t}$
- $\frac{dx}{dy} = \frac{1}{1+y^2}$

- $\frac{dy}{dt} = \sin 2t$

$$y = \int \sin(2t).dt$$

$$y = -\frac{1}{2}\cos(2t) + c$$

- $\frac{dy}{dx} = x^4 - 3x^2 + 2$

$$y = \int (x^4 - 3x^2 + 2).dx$$

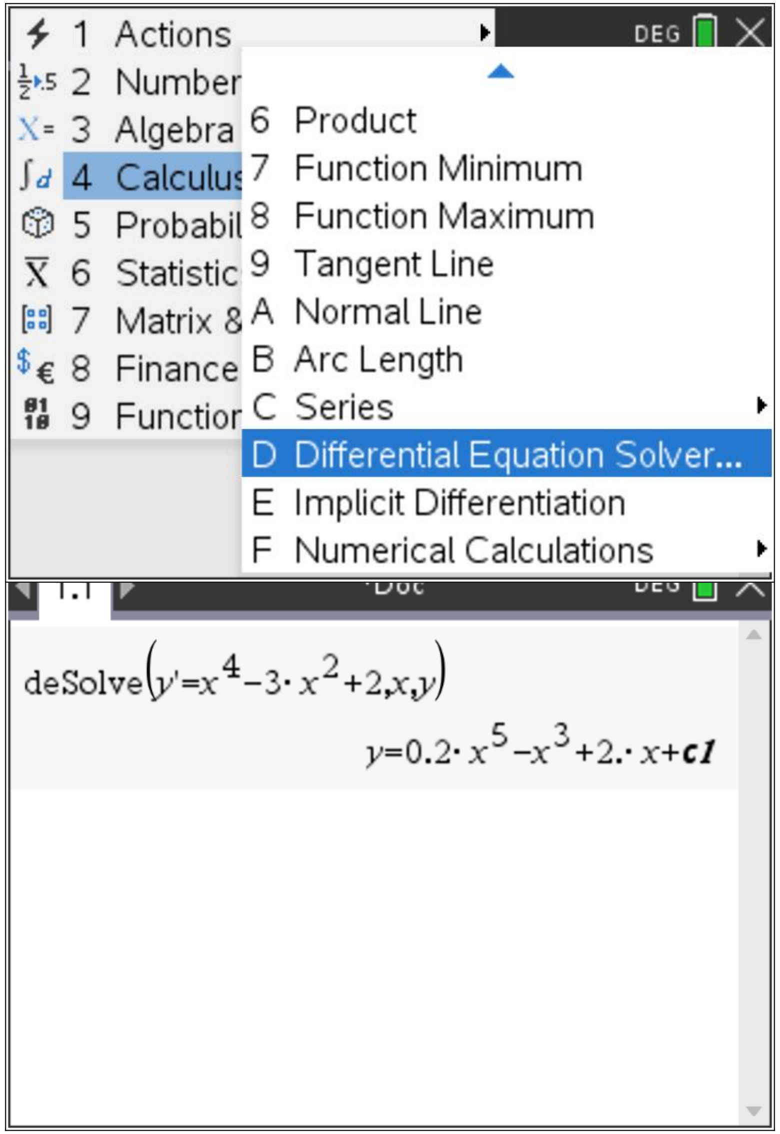
$$y = \frac{x^5}{5} - x^3 + 2x + c$$

Using the deSolver

Using the CAS for these equations can be a life saver! Take note of this for the VCAA exams at the end of the year. Using the CAS we can do the following:

Find the general solution of each of the following:

- $\dfrac{dy}{dx} = x^4 - 3x^2 + 2$



Starting with something simple

Find the general solution of each of the following:

- $\frac{dy}{dx} = x^4 - 3x^2 + 2$
- $\frac{dy}{dt} = sin(2t)$
- $\boldsymbol{\frac{dx}{dt} = e^{-3t} + \frac{1}{t}}$
- $\boldsymbol{\frac{dx}{dy} = \frac{1}{1+y^2}}$

$\log_e |t|$

- $x = \int \left(e^{-3t} + \frac{1}{t}\right) \cdot dt$

$= -\frac{1}{3}e^{-3t} + \ln|t| + c$

- $x = \int \frac{1}{1+y^2} \cdot dy$

$= \tan^{-1}(y) + c$

Families of solutions

It is important to remember that when we are integrating, when there are not conditions provided then we are finding a solution which will give a family of curves.

For example:

$$\frac{dy}{dx} = x^2$$

Then

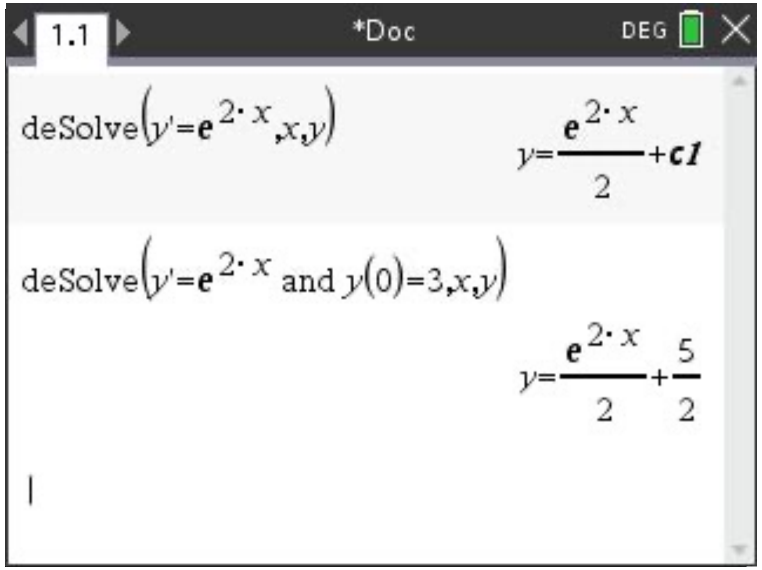
$$y = \frac{x^3}{3} + c$$

For a particular solution we need to be given something to enable us to find the value of c and hence narrow the answer down to one graph.

Example

- Find the family of curves with gradient given by e^{2x}. That is, find the general solution of the differential equation $\frac{dy}{dx} = e^{2x}$.
- Find the equation of the curve that has gradient e^{2x} and passes through (0,3).

$$y = \frac{1}{2}e^{2x} + c$$



Solving equations which require you to use two integrations

Find the general solution of each of the following:

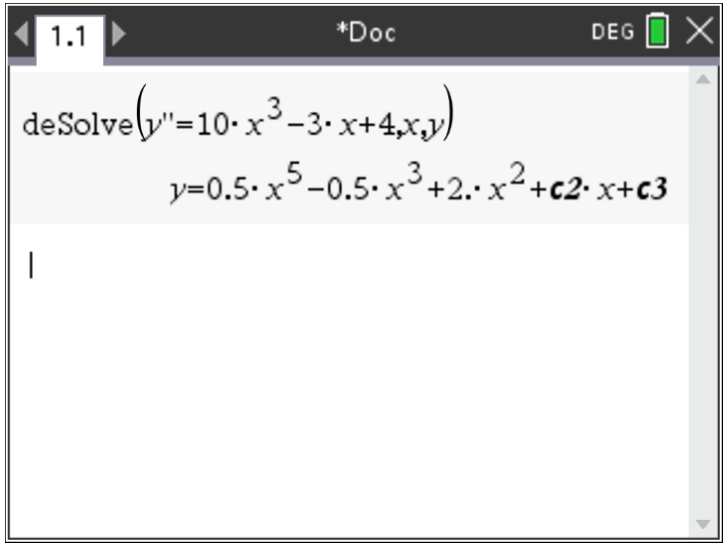
- $\frac{d^2y}{dx^2} = 10x^3 - 3x + 4$
- $\frac{d^2y}{dx^2} = \cos(3x)$
- $\frac{d^2y}{dx^2} = e^{-x}$
- $\frac{d^2y}{dx^2} = \frac{1}{\sqrt{x+1}}$

$$\frac{d^2y}{dx^2} = 10x^3 - 3x + 4$$

$$\frac{dy}{dx} = \frac{10x^4}{4} - \frac{3x^2}{2} + 4x + c$$

$$y = \frac{\cancel{10}x^5}{2\,\cancel{20}} - \frac{3x^3}{6} + \frac{4x^2}{2} + cx + d$$

$$= \frac{x^5}{2} - \frac{x^3}{2} + 2x^2 + cx + d$$



$$\frac{d^2y}{dx^2} = \cos 3x$$

$$\frac{dy}{dx} = \frac{1}{3}\sin 3x + c$$

$$y = -\frac{1}{9}\cos 3x + cx + d$$

Solving equations which require you to use two integrations

Find the general solution of each of the following:

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- $\frac{d^2y}{dx^2} = \cos(3x)$
- $\boldsymbol{\frac{d^2y}{dx^2} = e^{-x}}$
- $\boldsymbol{\frac{d^2y}{dx^2} = \frac{1}{\sqrt{x+1}}}$

$$\frac{d^2y}{dx^2} = e^{-x}$$

$$\frac{dy}{dx} = -e^{-x} + c$$

$$y = e^{-x} + cx + d$$

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{x+1}} = (x+1)^{-1/2}$$

$$\frac{dy}{dx} = 2(x+1)^{1/2} + c$$

$$y = \frac{4}{3}(x+1)^{3/2} + cx + d$$

Solving equations which require you to use two integrations

Consider the differential equation $\frac{d^2y}{dx^2} = \cos^2 x$.

- Find the general solution.
- Find the solution given that $\frac{dy}{dx} = 0$ when $x = 0$ and that $y(0) = -\frac{1}{8}$.

$\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$

$$\frac{d^2y}{dx^2} = \cos^2 x$$

$$\frac{dy}{dx} = \frac{1}{2}\int (\cos 2x + 1)\,dx$$

$$\frac{dy}{dx} = \frac{1}{4}\sin 2x + \frac{x}{2} + c$$

$$y = \int \left(\frac{1}{4}\sin 2x + \frac{x}{2} + c\right) dx$$

$$= -\frac{1}{4}\cdot\frac{1}{2}\cos 2x + \frac{x^2}{4} + cx + d$$

$$= -\frac{1}{8}\cos 2x + \frac{x^2}{4} + cx + d$$

Solving equations which require you to use two integrations

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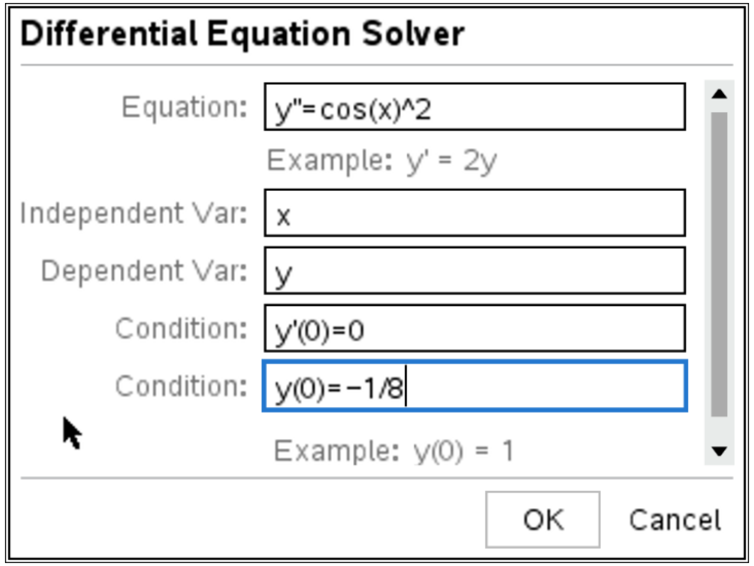
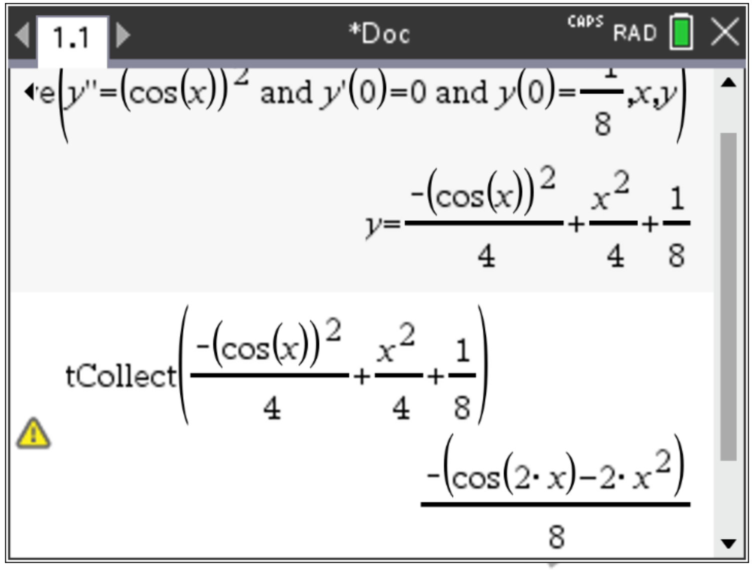
- Find the general solution.
- Find the solution given that $\frac{dy}{dx} = 0$ when $x = 0$ and that $y(0) = -\frac{1}{8}$.

$$y = \frac{-1}{8}\cos(2x) + \frac{1}{4}x^2$$

$=$





Learning Objectives: Reviewed

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Know how to solve differential equations of the form $\frac{dy}{dx} = f(x)$
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- Know how to solve differential equations of the form $\frac{d^2y}{dx^2} = f(x)$

Questions to complete

The following represents an indication of the minimum number of questions to complete for this exercise. If you choose to do more, then all good. Note that you should also aim to complete some questions from Chapter Reviews too.

Chapter 9B : Differential equations involving a function of the independent variable
Questions: 1adfhi, 2bdf, 3cfhj, 4efg, 5c, 6c