An introduction to differential equations

Year 12 Specialist Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Know what a differential equation is
- Understand how to verify a solution to a differential equation

Recap of past learning

This is the start of a new topic in Specialist Mathematics but isn't anything new in terms of content. We are going to look at what a differential equation is and how to solve them. This, to be honest, is really integration in a different form.

They are really useful and have practical real-world applications … so don't get lulled into a false sense of security!!

Differential equations contain derivatives of a particular function or relation.

The following are examples of differential equations:

$$\frac{dy}{dx} = \sin x$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y}{y+1}$$

General solutions

When we take the following and integrate it, we are finding a general solution which creates a family of functions.

$$\frac{dy}{dx} = \cos x$$

This becomes:

$$y = \int \cos x \, dx$$

Which becomes:

$$y = \sin x + c$$

To find a particular solution we require further information

Starting with something simple

We can verify that a particular expression is a solution of a differential equation by substitution.

- Verify that $y = Ae^x - x - 1$ is a solution of the differential equation $\frac{dy}{dx} = x + y$.
- Hence find the particular solution of the differential equation given that $y(0) = 3$.

$$\frac{dy}{dx} = x + y$$

$$Ae^x - 1 = \cancel{x} + Ae^x \cancel{- x} - 1$$

$$Ae^x - 1 = Ae^x - 1 \quad \checkmark$$

$$y = Ae^x - x - 1$$

$$3 = A.e^0 - 0 - 1$$

$$3 = A - 1$$

$$A = 4$$

$$y = Ae^x - x - 1$$

$$\frac{dy}{dx} = Ae^x - 1$$

$(0,3)$

$$\therefore y = 4e^x - x - 1$$

Another example

Verify that $y = e^{2x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

$y = e^{2x}$

$\frac{dy}{dx} = 2e^{2x}$

$\frac{d^2y}{dx^2} = 4e^{2x}$

$4e^{2x} + 2e^{2x} - 6(e^{2x}) = 0$

$6e^{2x} - 6e^{2x} = 0$

✓ ✓

Example

Verify that $y = ae^{2x} + be^{-3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0.$

$$y = ae^{2x} + be^{-3x}$$

$$\frac{dy}{dx} = 2ae^{2x} - 3be^{-3x}$$

$$\frac{d^2y}{dx^2} = 4ae^{2x} + 9be^{-3x}$$

$$\cancel{4ae^{2x}} + \cancel{9be^{-3x}} + \cancel{2ae^{2x}} - \cancel{3be^{-3x}} - \cancel{6ae^{2x}} - \cancel{6be^{-3x}} = 0$$

✓

Example

Find the constants a and b if $y = e^{4x}(2x+1)$ is a solution of the differential equation shown below:

$$\frac{d^2y}{dx^2} - a\frac{dy}{dx} + by = 0$$

$$y = e^{4x}(2x+1)$$

$$\frac{dy}{dx} = e^{4x}.2 + (2x+1)4e^{4x}$$

$$= 2e^{4x} + 4e^{4x}(2x+1)$$

$$= 2e^{4x}(1 + 2(2x+1))$$

$$= 2e^{4x}(4x+3)$$

$$\frac{d^2y}{dx^2} = 2e^{4x}(4) + (4x+3).8e^{4x}$$

$$= 32e^{4x}(x+1)$$

$$32\cancel{e^{4x}}(x+1) - 2a\cancel{e^{4x}}(4x+3) + b\cancel{e^{4x}}.(2x+1) = 0$$

$$32x + 32 - 8ax - 6a + 2bx + b = 0$$

$$x(32 - 8a + 2b) + 32 - 6a + b = 0$$

$$32 - 8a + 2b = 0$$

$$32 - 6a + b = 0$$

$$a = 8 \quad b = 16$$

Learning Objectives: Reviewed

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Questions to complete

The following represents an indication of the minimum number of questions to complete for this exercise. If you choose to do more, then all good. Note that you should also aim to complete some questions from Chapter Reviews too.

Chapter 9A : An introduction to differential equations
Questions: 1adf, 2cdeij, 3, 5, 7