

Lengths of curves in the plane

Year 12 Specialist Maths
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand how to find the length of curves in the place.



Recap of past learning

This is the final section of the course on applications of integration and it seems to be a section which features every year on both the Calculator and Non-calculator papers. Like revolutions around an axis, it's hard to imagine how this can get complex as you really need to use only one formula … which is on the Formula Sheet!

Spoiler Alert:

arc length	$\int_{x_1}^{x_2} \sqrt{1+\left(f'(x)\right)^2}\,dx$ or $\int_{t_1}^{t_2} \sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}\,dt$

The second formula is used when we have equations given in parametric form.

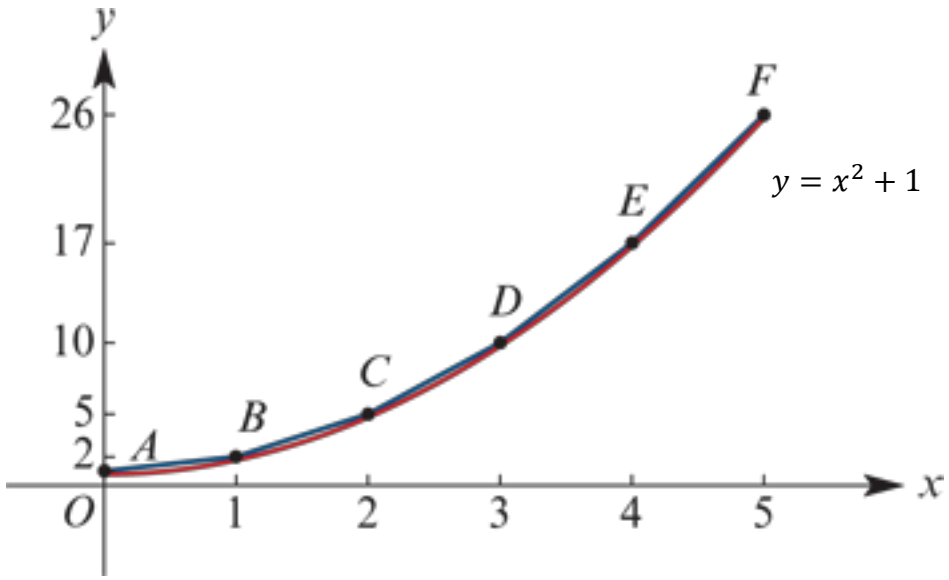


Back to basics with integration.

We know that integration is effectively taking very narrow rectangular strips between two limits, the curve and the x-axis and finding the area. The narrower the strips, the more “accurate” the area.

It makes sense that we can do exactly the same with the length of small sections of the curve.

We can define the length as the limit of a sum of lengths of line segments.





Back to basics with integration.

Taking a small portion we can see that the length of PQ (using Pythagoras' Theorem) is:

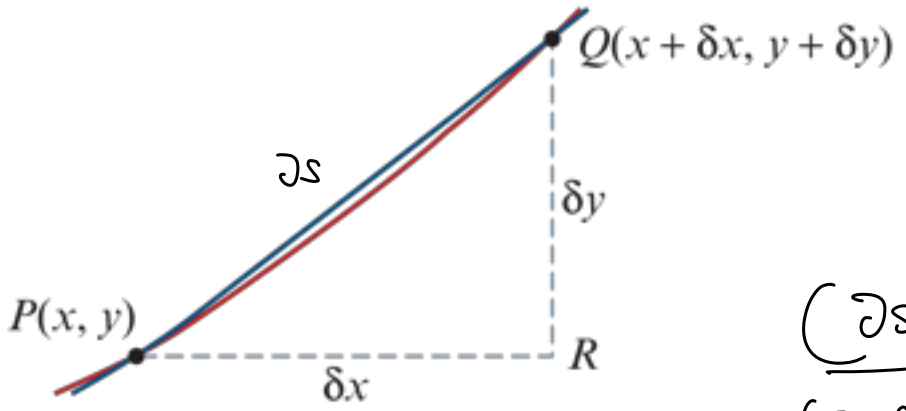
$$(\delta s)^2 \approx (\delta x)^2 + (\delta y)^2$$

Dividing each by $(\delta x)^2$ gives us:

$$\left(\frac{\delta s}{\delta x}\right)^2 \approx 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

Or ...

$$\delta s = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\, dx$$



$$\frac{(\partial s)^2}{(\partial x)^2} = \frac{(\partial x)^2}{(\partial x)^2} + \frac{(\partial y)^2}{(\partial x)^2}$$

If we then sum all these small pieces together between two limits, we can find the length of any curve using:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Which can be more expressed as:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$



Example

Find the length of the curve $y = x^{\frac{3}{2}}$ for $1 \leq x \leq 4$.

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

$$y = x^{3/2}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4}x$$

$$L = \int_1^4 \sqrt{1 + \frac{9}{4}x}\,.\,dx$$

$$= \int_1^4 \left(\frac{9}{4}x + 1\right)^{1/2}.\,dx$$

$$u = \frac{9}{4}x + 1$$

$$\frac{du}{dx} = \frac{9}{4}$$

$$dx = \frac{4}{9}\,.du$$

$$\frac{4}{9}\int_{\frac{13}{4}}^{10} u^{1/2}\,.\,.\,du$$

$$L = \frac{4}{9}\left[\frac{2u^{3/2}}{3}\right]_{13/4}^{10} = \frac{4}{9}\left(\frac{20\sqrt{10}}{3} - \frac{13\sqrt{13}}{12}\right)$$

$$= \frac{80\sqrt{10} - 13\sqrt{13}}{27}$$



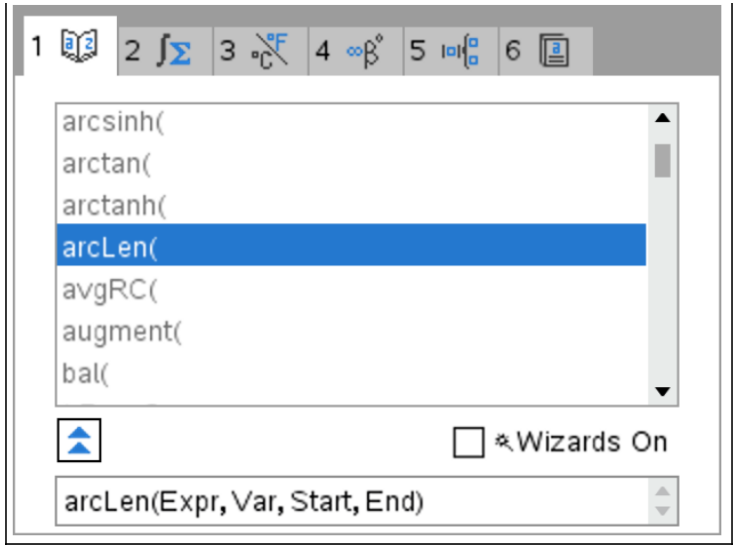
Note this could also have been asked as:

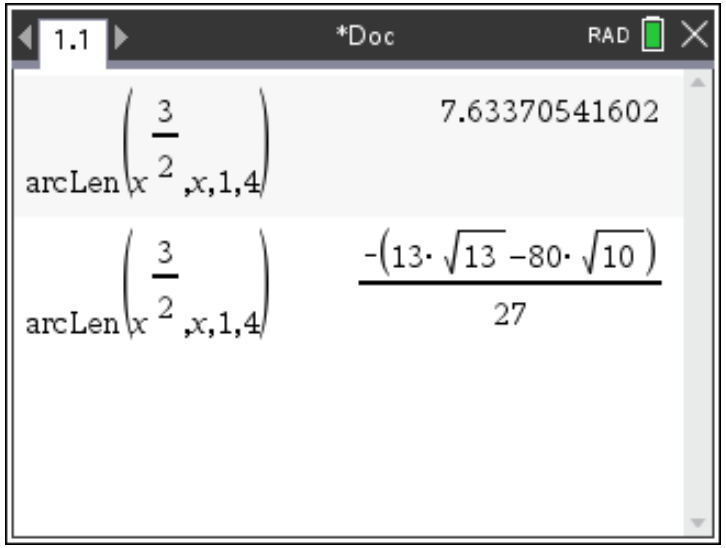
Find the length of the curve defined by the parametric equations:

$x = t$ and $y = t^{\frac{3}{2}}$ for $1 \leq t \leq 4$

Example: Using the CAS

Find the length of the curve $y = x^{\frac{3}{2}}$ for $1 \leq x \leq 4$.







Length of a parametric curve

As there will be a time when we will review the work on parametrics again, it's important to note that we can find the length of a curve using parametric equations.

So, going back to the previous theory:

$$(\delta s)^2 \approx (\delta x)^2 + (\delta y)^2$$

If we now divide each by $(\delta t)^2$ gives us:

$$\left(\frac{\delta s}{\delta t}\right)^2 \approx \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$$

Or

$$\delta s \approx \sqrt{\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2}\, dt$$

Which can be more expressed as:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$\frac{dx}{dt} = 1$

$x = t \qquad y = t^{3/2} \qquad \frac{dy}{dt} = \frac{3}{2}t^{1/2}$

$\left(\frac{dy}{dt}\right)^2 = \frac{9t}{4}$

$$L = \int_1^4 \sqrt{1 + \frac{9}{4}t}\, dt$$

$$= \frac{1}{2}\left[\frac{2(4+9t)^{3/2}}{27}\right]_1^4$$

$=$ ...



Example: Length of a parametric curve

Find the length of the curve defined by the parametric equations $x = \cos t$ and $y = \sin t$, for $0 \leq t \leq 2\pi$.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$x = \cos t$

$\frac{dx}{dt} = -\sin t$

$\left(\frac{dx}{dt}\right)^2 = \sin^2 t$

$y = \sin t$

$\frac{dy}{dt} = \cos t$

$\left(\frac{dy}{dt}\right)^2 = \cos^2 t$

$$L = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, . dt = \int_0^{2\pi} 1 . dt$$

$$L = \left[t \right]_0^{2\pi}$$

$L = 2\pi$ units



Learning Objectives: Revisited

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