Area of a region between two curves

Year 12 Specialist Maths
Units 3 and 4

www.maffsguru.com

Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand how to find the area of a region between two curves.

Recap of past learning

This work once again builds on the work covered in the Methods 1 and 2 and Methods 3 and 4 course. It requires you to understand that you are effectively finding the area under two separate curves and then subtracting them.

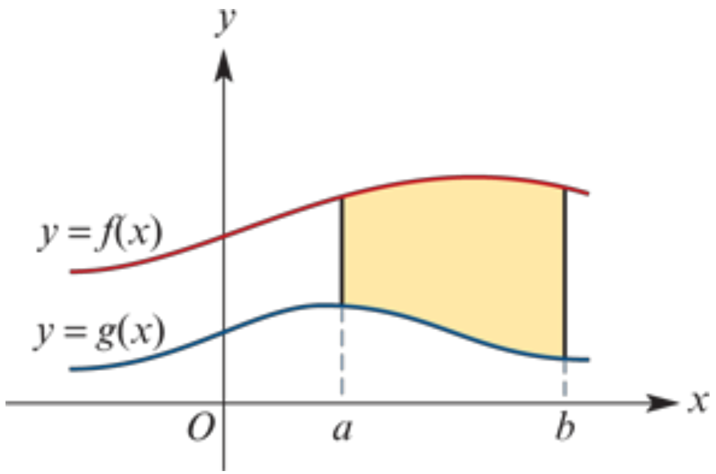
You will be expected to use the more complex integration work we have covered in Specialist Mathematics Units 3 and 4.

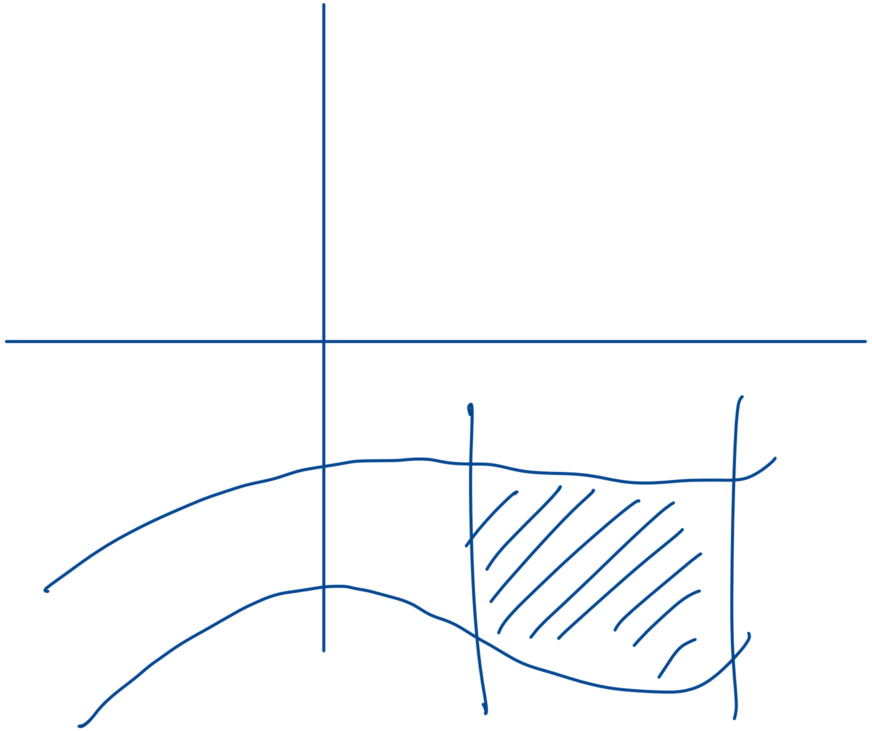
Examples have been extracted, with permission, from the Cambridge Mathematical Methods Units 1 and 2 Textbook

www.maffsguru.com

Finding the area under two curves

What is important to note here is that the concept of signed area does not apply to this topic and can cause confusion. It is important to note that if we think of translating both functions above the x-axis (when any section is below) that we will still have the same area!



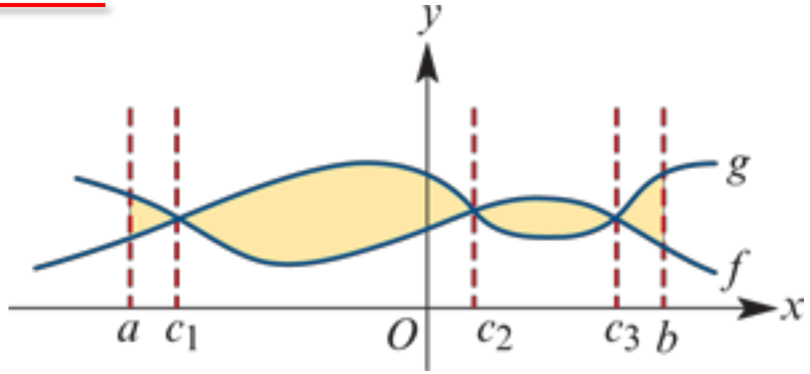


Points of intersection

When being asked to find the total area under two functions it’s important to note where the functions may intersect (or pass above and below each other).

For all functions, you are always looking to find the difference in area between the highest function (at a point) and the lowest function (at a point).

The following example would need to have four separate integrals considered.



Note: It’s important to sketch the functions first and then find points of intersection. Using the CAS is the quickest way!

Example

Find the area of the region bounded by the parabola $y = x^2$ and the line $y = 2x$.

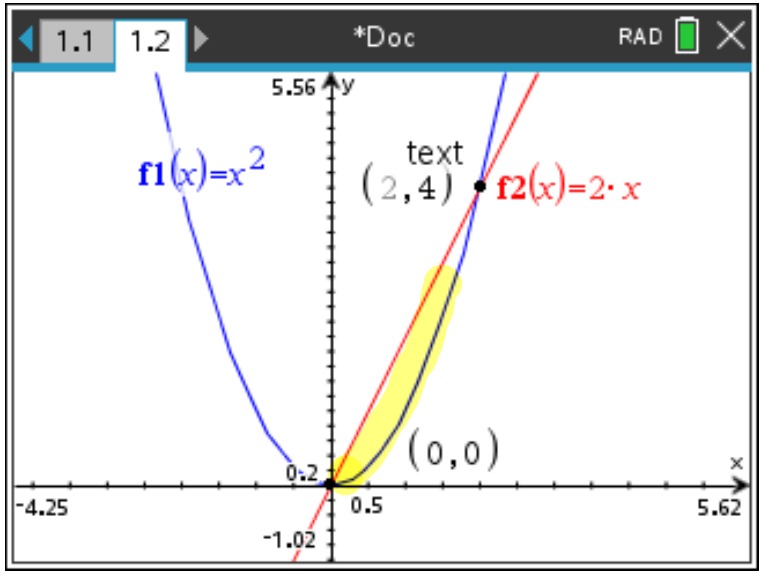
$$A = \int_0^2 \left(f_2(x) - f_1(x)\right)\,dx$$

$$= \int_0^2 \left(2x - (x^2)\right)\,dx$$

$$= \int_0^2 \left(2x - x^2\right)\,dx$$

$$= \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^2$$

$$= \left(4 - \frac{8}{3}\right) = \frac{12}{3} - \frac{8}{3} = \frac{4}{3}$$



$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

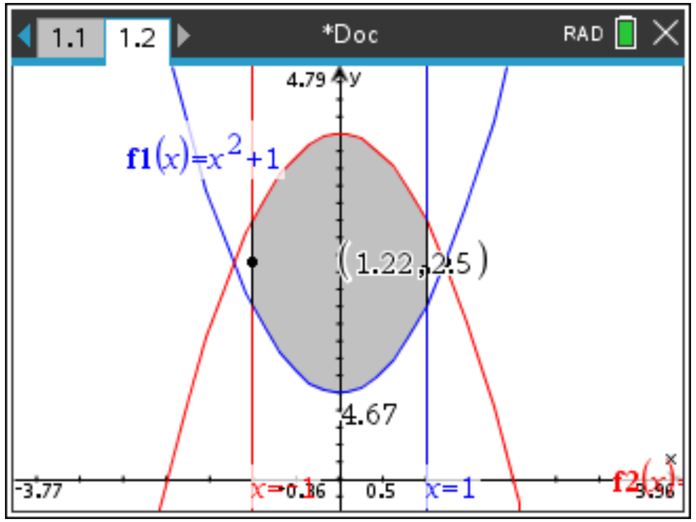
$$x = 0 \quad x = 2$$

Examples have been extracted, with permission, from the Cambridge Mathematical Methods Units 1 and 2 Textbook

www.maffsguru.com

Example

Calculate the area of the region enclosed by the curves with equations $y = x^2 + 1$ and $y = 4 - x^2$ and the lines $x = -1$ and $x = 1$.



$$A = \int_{-1}^{1} \left((4 - x^2) - (x^2 + 1) \right) dx$$

$$= \int_{-1}^{1} (4 - x^2 - x^2 - 1)\, dx$$

$$= \int_{-1}^{1} (3 - 2x^2)\, dx$$

$$= \left[3x - \frac{2x^3}{3} \right]_{-1}^{1}$$

$$= \left(3 - \frac{2}{3}\right) - \left(-3 + \frac{2}{3}\right)$$

$$= 3 - \frac{2}{3} + 3 - \frac{2}{3}$$

$$= 6 - \frac{4}{3}$$

$$= \frac{18}{3} - \frac{4}{3} = \frac{14}{3}$$

Example

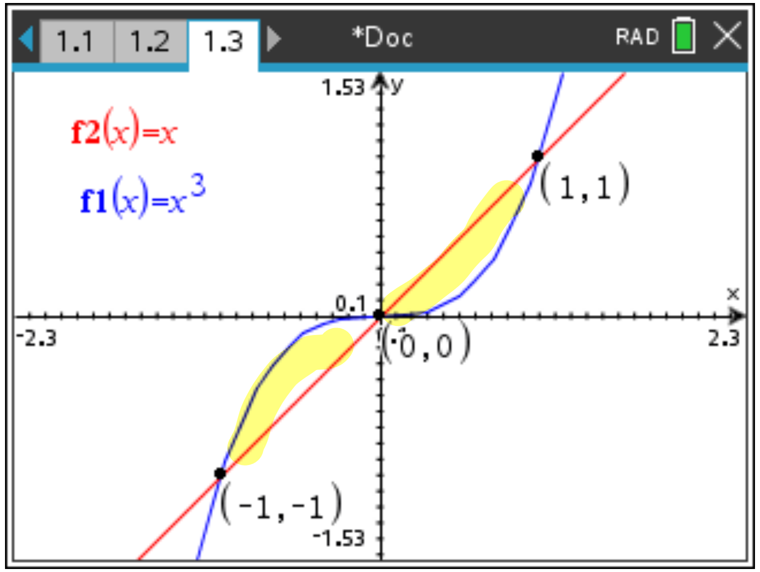
Find the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$.

$$A = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx$$

$$A = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1}$$

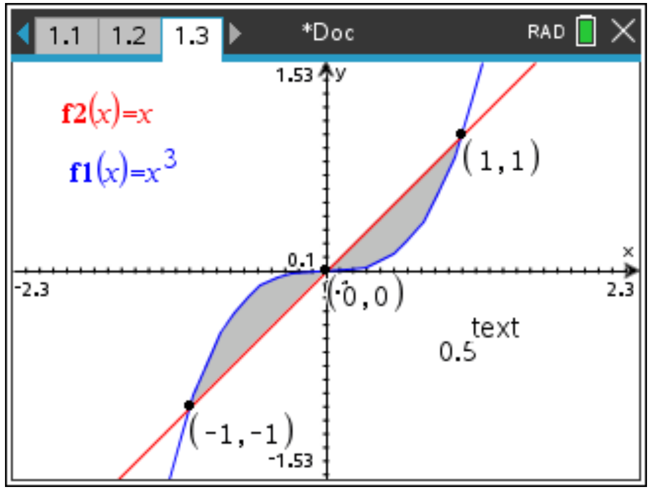
$$= 0 - \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) - 0$$

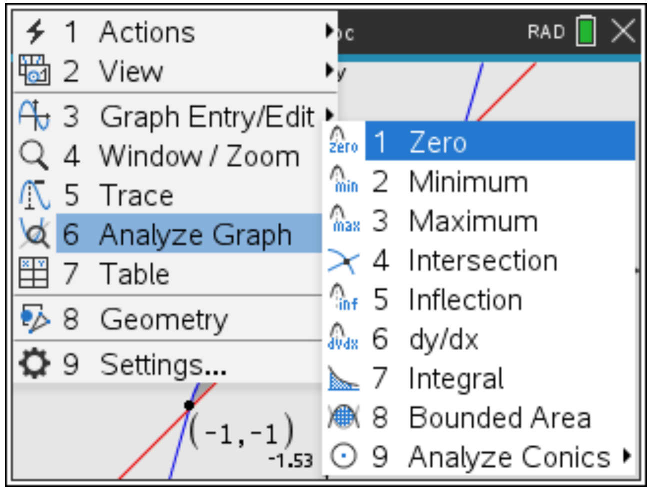
$$= \frac{1}{4} + \frac{1}{4} = \underline{\underline{\frac{1}{2}}}$$



Using the CAS

Find the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$.

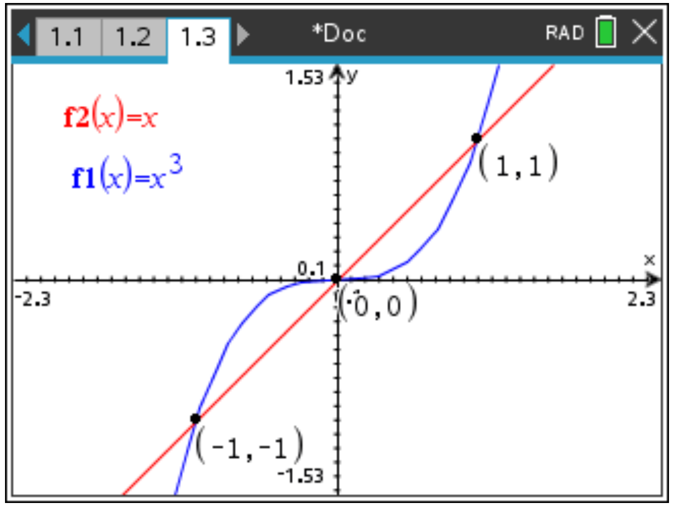


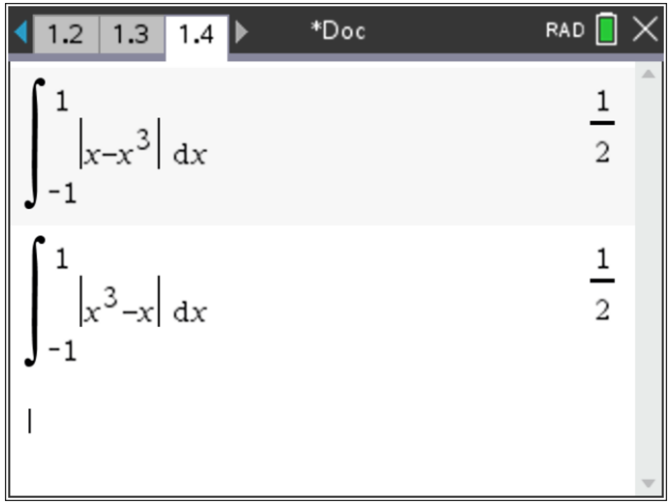


Using the absolute value function

Find the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$.

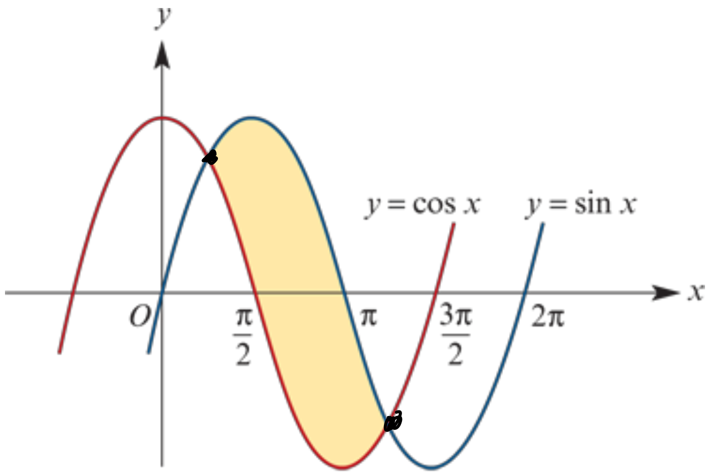
$$\int_{-1}^{1} |x - x^3|\, dx$$





Example

Find the area of the shaded region.



$\sin x = \cos x$

$\tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx$$

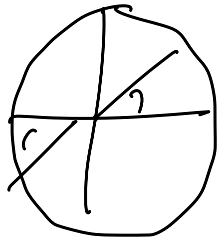
$$A = \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$A = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= 2\sqrt{2}$$

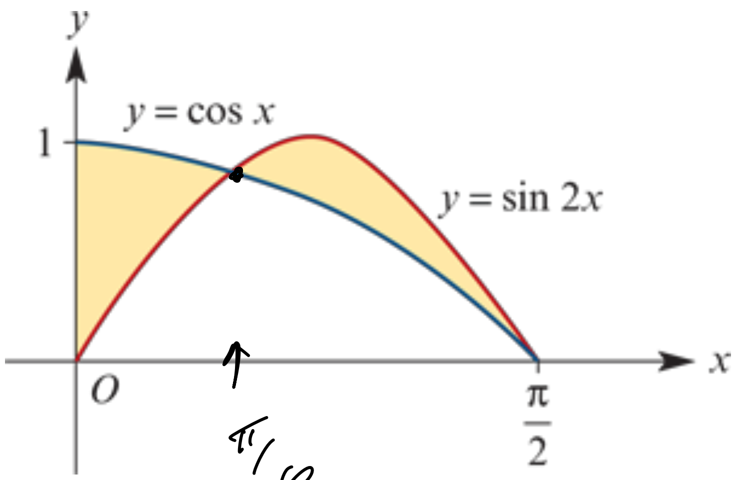
$\sin \rightarrow \cos$

$\cos \rightarrow -\sin$

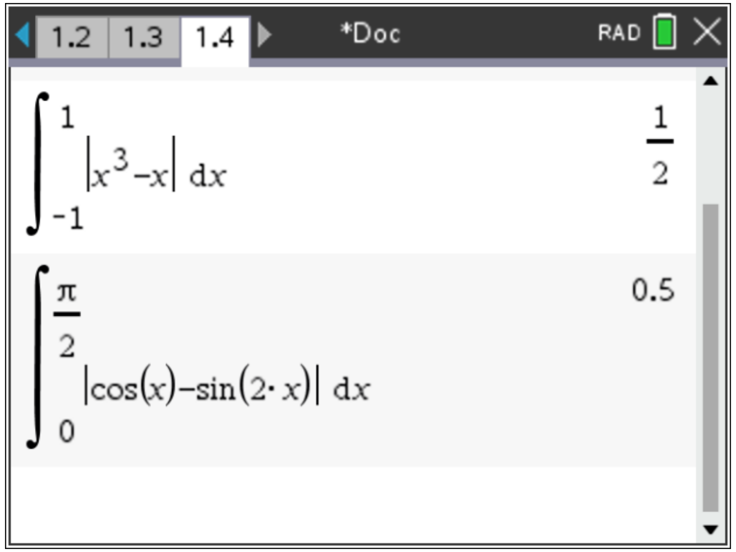


Example

Find the area of the shaded region.



$\pi/6$



$\cos x = \sin 2x$

$$A = \int_0^{\pi/6} (\cos x - \sin 2x)\, dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\, dx$$

$$= \frac{1}{2}$$

$\cos x = 2\sin x \cos x$

$2\sin x \cos x - \cos x = 0$

$\cos x (2\sin x - 1) = 0$

$\cos x = 0 \qquad \sin x = \frac{1}{2}$

$x = \frac{\pi}{2} \qquad \frac{\pi}{6}$

Learning Objectives: Revisited

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand how to find the area of a region between two curves.

Questions to complete

The following represents an indication of the minimum number of questions to complete for this exercise. If you choose to do more, then all good. Note that you should also aim to complete some questions from Chapter Reviews too.

Chapter 10B : Area of a region between two curves
Questions: 1, 3, 5, 7cde, 8, 11, 12, 13, 16, 17